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On the Analytical Solutions of the Oblique Shock Wave Equation

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Original Research Article

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Abstract: The present paper presents algebraic, analytical, approximate and iterative solutions and characteristics of oblique shock wave equation in a supersonic freestream. A closed-form solution of the cubic polynomial equation is having real and conjugate complex roots and associates with strong and weak shock wave. For known initial conditions and shock wave angle, upstream conditions can be obtained through careful application of the oblique shock wave table. But a numerical method is convenient as compared to oblique shock wave's graphs or tables for preliminary designing of high-speed vehicle. A numerical algorithm is developed based on the closed-form solution that can easily employ for rapid calculations of oblique shock wave angle. A computer program is written in FORTRAN language and compiled and executed on Linux to compute the oblique shock wave angle β. The exact solution of the oblique shock equation in terms of $tan\beta$ is obtained to compute oblique shock wave angle for given upstream flow conditions. The objective of the present study is to numerically evaluate shock wave angle and for the known entering Mach number.
Keywords: algebraic, FORTRAN language, angle β, supersonic freestream.

INTRODUCTION

The oblique shock wave theory essentially requires for designing a wave rider configuration and an air-breathing of high-speed vehicle [1]. The oblique shock wave angle β is needed as an explicit function of the upstream Mach number M_1 and the flow deflection angle θ. It is commonly known as θ-β-M relationship of the oblique shock wave theory associated with compressible gas-dynamics. Ames Research Staff [2] has mentioned in their report that the analytical solution of the oblique shock equation cannot be arrived and tabulated values of θ-β-M.

Thompson [3] has derived a cubic polynomial equation in term of $sin\beta$ of the oblique shock wave equation. Briggs [4] and Mascitti [5] have obtained analytical solution of the cubic equation. Naylor [6] has presented a solution of the shock-wave cubic equation that allows computation of the oblique shock wave angle without tables. Hartley et al. [7] have carried out real-time application of the exact and approximate solutions to the oblique shock wave equations.

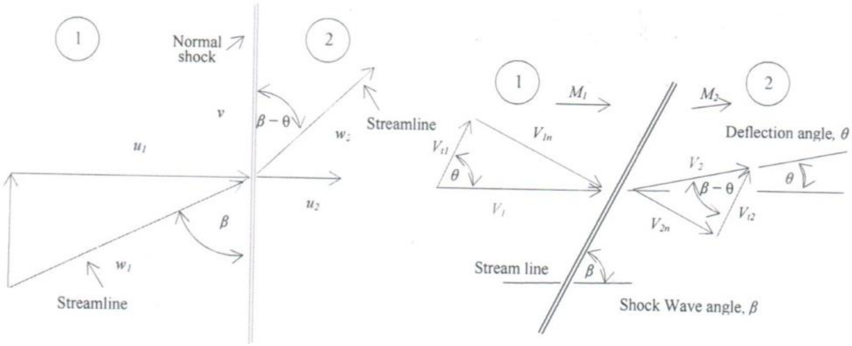
The cubic polynomial equation of the oblique shock wave equation in terms of $tan\beta$ had been derived by Wellmann [8] and the analytical solutions were published by Wolf [9]. An analytical solution was also obtained by Emanuel [10] and compared with the tabulated value of β by Anderson [11]. Bar-Meir [12] has discussed in detail the characteristics of the real and complex conjugate roots of the cubic polynomial equation of the oblique shock wave.

Duo *et al*. [13], Powers [14] and Agnone [15] have discussed the approximate formula for weak and strong shock wave angles. Houghton and Brock [16] and Houghton and Carpenter [17] presented iteration method for the solution of cubic polynomial equation. Rudd and Lewis [18] have compared the closed-form solutions with the iterative scheme [16]. They concluded that the computer algorithm of the iterative method is too complicated and consuming more computer time as compared to the analytical solution.

The above literature survey reveals that analytical, approximate and iterative methods are available to obtain the value of the oblique shock wave angle for given upstream flow conditions. The paper presents a closed-form solution for the shock wave angle based on the oblique shock wave theory. The roots are obtained using Cardan cubic polynomial equation [19, 20]. A computer program is developed based on the exact solution of the oblique shock theory which can easily implement in the preliminary design phase of an air-breathing and a wave rider high-speed vehicle.

Analytical Solution

Figure 1 (a) delineates incoming streamline w_1 with an angle β and outgoing streamline w_2 with makes an angle ($\beta - \theta$) with the normal shock. The oblique shock can be considered as being formed by superimposing a flow parallel to the normal shock as shown in Fig. 1 (a) and remains constant as depicted in Fig. 1 (b). The velocities components V_{t1} and V_{t2} are parallel ahead and behind the normal and oblique shock wave, respectively. Using Rankine-Hugonoit relations [11], $V_{t1} = V_{t2}$ and remain constant for stationary shock. It makes the angle β with respect to the oblique shock wave and θ represents the flow deflection angle. u_1 and u_2 are velocity components perpendicular to the normal shock as depicted in Fig. 1 (a). The resultant velocities V_1 and V_2 are ahead and behind the oblique shock wave.



(*a*) (*b*)

Fig-1: Flow through an oblique shock wave (*a*) normal shock (*b*) oblique shock

The density ratio (ρ_2/ρ_1) across the normal shock is

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} = \frac{(\gamma+1)}{\left(\frac{2}{M_1^2}\right)+(\gamma-1)} \quad (1a)$$

Subscripts 1 and 2 refer side on which flow quantities to be evaluated and depicted in Fig.1 in a circle, γ is the ratio of specific heats. Substituting $M_1 sin\beta$ for M_1 in Eq. (1a) to obtain the relation of flow variables for the oblique shock. Thus, all of the normal shock wave relations [11] can be converted to the oblique shock relation with substitution $M_1 sin\beta$, $M_2 \rightarrow M_2 sin\beta$ and obtain the ratio of the density ratio across the oblique shock as

$$\frac{\rho_2}{\rho_1} = \frac{\tan\beta}{\tan(\beta-\theta)} \quad (1b)$$

For small flow deflection angle θ it becomes

$$M_1^2 \sin^2\theta - 1 \approx \left(\frac{\gamma+1}{2} M_1^2 \tan\beta\right)\theta \quad (1c)$$

The relationship between the oblique shock wave angle β, the flow deflection angle θ across the oblique shock wave and upstream Mach number M_1 [21] is

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{(\gamma-1)M_1^2 \sin^2\beta + 1}{(\gamma+1)M_1^2 \sin^2\beta} \quad (2)$$

The above equation shows an implicit relation between θ-β-M. Eq. (2) becomes zero at $\beta = \pi/2$ and at $\beta = \mu = sin^{-1}(1/M_1)$, where μ denotes Mach wave angle. Within this range β is positive and must therefore have a maximum value of flow deflection angle θ_{max}. There are two ways to express the polynomial equation of Eq. (2). Thompson [3] has obtained following expression for Eq. (2) as

$$\sin^6\beta + b\sin^4\beta + c\sin^2\beta + d = 0 \quad (3)$$

where

$$b = -\left[\frac{M_1^2 + 2}{M_1^2} + \gamma \sin^2\theta\right]$$

$$c = \frac{2M_1^2 + 1}{M_1^4} + \left[\frac{(\gamma+1)^2}{4} + \frac{\gamma-1}{M_1^2}\right]\sin^2\theta$$

$$d = -\frac{\cos^2\theta}{M_1^4}$$

The Cardan equations yield three roots for $sin^2\beta$ from the foregoing equation. The direct computation of oblique shock wave properties with freestream Mach number and flow deflection angle as the independent variables is used to determine the strong and the weak shock wave angle. One of the root of the cubic equation is real and the solutions of Eq. (3) may written as

$$\beta_S = \tan^{-1}\left(\sqrt{\frac{x_S}{1-x_S}}\right) \quad (4a)$$

$$\beta_W = \tan^{-1}\left(\sqrt{\frac{x_W}{1-x_W}}\right) \quad (4b)$$

where

$$v = \frac{\left(3d - b^2\right)}{9}, \quad w = \frac{\left(9bc - 27d - 2b^2\right)}{54}, \quad D = v^3 + w^2$$

$$x_S = -\frac{b}{3} + 2\sqrt{-v}\cos\phi, \quad x_W = -\frac{b}{3} - \sqrt{-v}\left(\cos\phi - \sqrt{3}\sin\phi\right), \quad \phi = \frac{1}{3}\left(\tan^{-1}\frac{\sqrt{-D}}{w} + \Delta\right)$$

Where subscripts s and w represent the strong and the weak shock, respectively. If $\Delta = 0$ then $w \geq 0$; and if $\Delta = \pi$, then $w < 0$. Normal shock wave occurs just at $\theta = 0$ and $\beta = \pi$. If $\theta > \theta_{max}$, then no solution exists for a straight oblique shock wave. If $\theta < \theta_{max}$, then there are two values of β for a given value of M_1. The large value gives a strong shock solution where downstream M_2 is subsonic. The small value gives the weak shock solution where M_2 is supersonic except for a small region near θ_{max}. In nature, the weak shock solution is favored and that one usually occurs.

By writing another general form of Eq. (2) in a cubic relation for $tan\beta$, Wellmann [8] derived following equation

$$\tan\theta\left[\left(\frac{\gamma-1}{2}\right) + \left(\frac{\gamma+1}{2}\right)\tan^2\alpha\right]\tan^3\beta - \tan^2\beta + \tan\theta\left[\left(\frac{\gamma+1}{2}\right) + \left(\frac{\gamma+3}{2}\right)\tan^2\alpha\right]\tan\beta + \tan^2\alpha = 0 \quad (5)$$

Let us introduce a new variable $tan\beta = y$ and Eq. (5) becomes

$$y^3 + by^2 + cy + d = 0 \quad (6)$$

where

$$b = \left(\frac{\frac{2}{\gamma-1}\frac{\cos\theta}{\sin\theta}}{M_1^2 + \frac{2}{\gamma-1}}\right), \quad c = \left(\frac{\frac{\gamma+1}{\gamma-1}M_1^2 + \frac{2}{\gamma-1}}{M_1^2 + \frac{2}{\gamma-1}}\right), \quad d = \left(\frac{\frac{2}{\gamma-1}\frac{\cos\theta}{\sin\theta}}{M_1^2 + \frac{2}{\gamma-1}}\right)\left(1 - M_1^2\right)$$

Let us introduce another variable say $y = x - b/3$ in Eq. (6) and reduced to

$$x^3 + vx + w = 0 \quad (7)$$

The three roots of Eq. (7) are as following

$$x_1 = A + B \tag{8a}$$

$$x_{2,3} = -\frac{1}{2}(A+B) \pm \frac{\sqrt{3}}{2}(A-B)i \tag{8b}$$

where

$$A = \left[-\frac{w}{2} + (D)^{1/2}\right]^{1/3}, \quad B = \left[-\frac{w}{2} - (D)^{1/2}\right]^{1/3}, \text{ and } D = \frac{w^2}{4} + \frac{v^3}{27}$$

If $D > 0$, then one real root of Eq. (6) is

$$\beta = \tan^{-1}\left\{\left(A+B\right) - \frac{b}{3}\right\} \tag{9a}$$

and the other roots depend on the magnitude of *D*. They can be expressed as

$$\beta_1 = \tan^{-1}\left\{-2B_1 \cdot \cos\left[\cos^{-1}\left(\frac{\phi}{3}\right)\right] - b\right\} \tag{9b}$$

$$\beta_2 = \tan^{-1}\left\{-2B_1 \cdot \cos\left[\cos^{-1}\left(\frac{\phi}{3}\right) + \frac{\pi}{2}\right] - b\right\} \tag{9c}$$

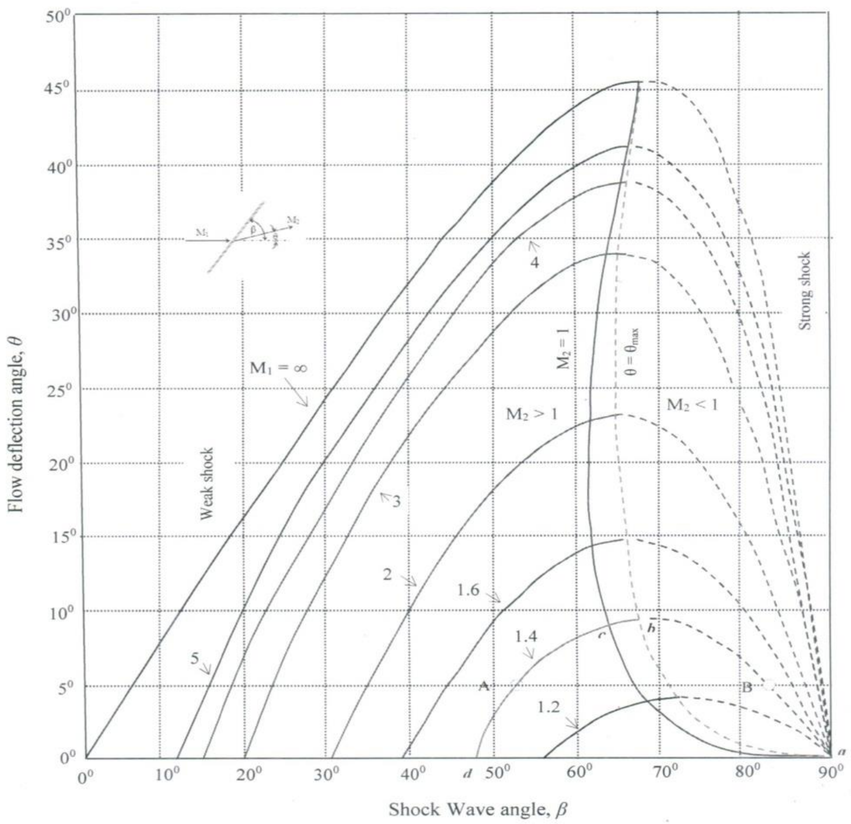
$$\beta_3 = \tan^{-1}\left\{-2B_1 \cdot \cos\left[\cos^{-1}\left(\frac{\phi}{3}\right) - \frac{\pi}{2}\right] - b\right\} \tag{9d}$$

where

$$B_1 = \sqrt{-\frac{v}{3}}, \quad A_1 = -\frac{w}{2}\frac{1}{B_1^3} \quad \text{and} \quad \phi = \cos^{-1}(A_1)$$

A numerical algorithm is written based on the magnitude of *D*. A computer program can be integrated for preliminary aerodynamic design and compiled in Linux or UNIX complier. The computer subroutine will compute the real root, the weak and the strong shock wave angle with the specific upstream Mach number and the flow deflection angle. The quadratic equation is also employed to solve the flow over a spinning sphere in ideal flow [22].

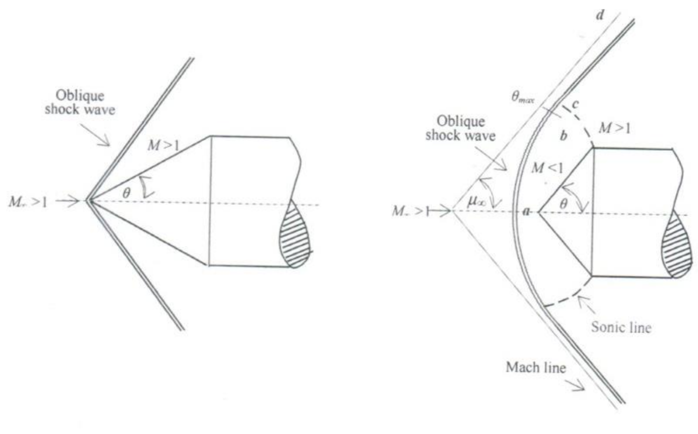
Figure 2 is drawn with the help of Ref. [2] and can also obtain with the present numerical algorithm. The oblique shock wave shows several interesting characteristics that depend on values of β, θ and *M*. The point *a*, where the shock angle equals the Mach angle, is described by the equations of acoustics. On *ac* shock is said to be weak (dotted curve), and on cd it is said to be strong (solid line). At *b*, the $\theta = \theta_{max}(M_1)$ is the maximum possibly by the oblique shock alone.



Shock alone

Fig-2: Variation of shock wave angle with flow-deflection angle for various freestream Mach number [*a* (π/2), *b* (θ_{max}), *c* (M = 1), *d* (μ_∞)]

On *ab*, the flow downstream of the shock is $M_2 < 1$. On *bd*, it is $M_2 > 1$. Point *b* is always to the left of *c* and very close to it. The value of β never differs by more than 4.5 deg., nor do the values of θ differ by more than 0.5 deg. Therefore, the flow through a weak shock nearly always keeps the speed of supersonic, and that through a strong shock always makes the speed subsonic. Since the range of θ angles for which a weak shock has subsonic flow behind it is less than 0.5 deg. A normal shock is always a strong shock; it corresponds to point *d* ($M_2 < 1$). Weak and strong shocks are distinguished by dotted and continuous lines in Fig. 2.



(*a*) (*b*)

Fig-3: Flow through an oblique shock wave (a) attached shock (b) detached shock

[*a* (π/2), *b* (θ_{max}), *c* (M = 1), *d* (μ_1)]

For any given M_1, and $\theta < \theta_{max}$, there are two β's. The larger β is called strong shock solution, where M_2 is subsonic. Two such points are labeled *A* and *B*. one of these *A* is associated with higher shock angle and thus has a higher normal Mach number, which means that it is stronger shock with a resulting higher-pressure ratio. The other *B* has a lower shock angle and thus is a weaker shock with a lower pressure rise across the shock. The lower β is called the weak shock solution, where M_2 is supersonic except for a small region near θ_{max}. If $\theta = 0$, then $\beta = \pi/2$ (normal shock) or $\beta \to \mu = \sin^{-1}(1/M_1)$ (Mach wave) and $\beta_{min} = \mu$.

For β at large Mach number $M_1 \to \infty$ (θ then is small anyway) and $\beta \to \frac{\gamma+1}{2}\theta$.

For $\gamma = 1$, $\rho_2/\rho_1 \to \infty$ and $\tan(\beta - \theta) = 0$ or $\beta = \theta$

For $\gamma = 1.4$, $\rho_2/\rho_1 \to 6$ [Eq. 1 (b)] and $\tan\beta = \beta$ and $\tan(\beta - \theta) = \beta - \theta$, or $\beta = 1.2\,\theta$

If the θ is reduced to zero, we find the Mach angle μ_∞ as

$$\theta\big|_{\theta\to 0} = \mu_\infty = \sin^{-1}\left(\frac{1}{M_1}\right) \tag{10}$$

The maximum possible flow deflection θ at an oblique shock is 45.6 deg. and corresponding shock angle β is 67.8 deg. for diatomic gas ($\gamma = 1.4$). Thus, at the Mach angle $\mu = \sin^{-1}(1/M_1)$, the gradient is

$$\frac{d\beta}{d\theta} = \frac{4\left(M_1^2 - 1\right)}{(\gamma+1)M_1^2} \tag{11}$$

and deflection as $M_1 \to \infty$, it is

$$\frac{d\beta}{d\theta} = \frac{0.5(\gamma+1) - \gamma\sin^2\theta}{0.25(\gamma+1)^2 - \gamma\sin^2\theta} \tag{12}$$

Shocks that are not at right angles to the oncoming flow are said to be oblique flows at $M_1 > 1$. The upstream flow is necessarily $M_1 > 1$ but the downstream flow may be either $M_2 < 1$ or $M_2 > 1$, depending on the values of M_1 and β. An oblique shock may be attached or detached as shown in Fig. 3 (a) and (b), respectively. The upstream Mach number is M_1 and the downstream is M_2. The angle between the upstream flow and the shock is the shock angle β and the angle between the upstream and downstream is the deflection angle θ. For any given M_1, there is a maximum θ beyond which the shock will be curved and detached. It can be observed from Fig. 3 (b) that at point (*a*) the flow is perpendicular to the shock wave and the properties of the flow governed by the normal shock relations. In moving from point (*a*) to (*b*), the shock weakens and the deflection of the flow behind the shock increases until a point of maximum flow deflection θ_{max} is reached at (*b*). The Mach number behind the shock is subsonic up to point (*c*) where the Mach number just downstream of the shock is one. For the sake of clarity, we have marked symbols *a*, *b*, *c* and *d* which represent as *a* ($\pi/2$), *b* ($\theta_{\max}$), *c* ($M = 1$), *d* (μ_1) in Fig.2 and Fig. 3 (b).

Iterative Solution

An iterative solution [23] for the cubic equation can also be used to calculate β. For any root $(\tan\beta)_k$ is written with respect to the coefficient of the polynomial of Eq. (6) as

$$|y|_k \le \sqrt{b^2 - 2a} \tag{13}$$

Where subscript *k* is the root of Eq. (6). This gives an upper bound for all the roots of the polynomial equation. All the coefficients of the polynomial equation are real. Then all the roots of the equation are real or the pairs of roots are complex conjugates. The Newton-Raphson method is simple to calculate the real root of the cubic equation. The iteration yields the real root $y_1 = \tan\beta$ at $k = 1$. Eq. (6) reduced the following quadratic equation

$$y^2 + (b - y_1)y + (d - y_1) = 0 \tag{14}$$

Solution of Eq. (11) can be obtained using following formula

$$y = -(b - y_1) \pm \frac{\sqrt{(b - y_1)^2 - 4(d - y_1)}}{4} \quad (15)$$

The real root can be obtained by solving Eq. (6). The iteration scheme can be initiated with an initial value using Eq. (13). Other roots can be calculated by solving Eq. (15). The iterative scheme requires more computer time to obtain the oblique shock wave angle for the specific flow conditions and deflection angle.

A similar flow features (attached with supersonic speed behind the oblique shock, detached normal shock, with a subsonic pocket behind it) function of θ can be found at a ramp and blunt body. The entropy rises across them, wave drag which is a form of wave drag. The entropy rise is largest where the shock is normal to the freestream flow (normal shock), and becomes smaller with decreasing inclination of the shock against the freestream (oblique shock).

CONCLUSION

The paper compares the relative performance of analytical, approximate and iterative solutions of the oblique shock wave equation. The strong and weak shock wave angle can be calculated from the closed-form solution for given upstream flow conditions. The analytical solutions are useful and would lead to saving in computer time. A computer program for oblique shock wave is included that can be integrated for preliminary aerodynamic design. The numerical algorithm is efficient, simple and straightforward to implement in designing air-breathing and a wave rider at high-speed vehicle.

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