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# Background of Second-Order Linear Differential Equations

**Alaa Abalgaduir[1*], Ehssan Ahmed[2], Asma Mohamed[3], Shima Alwasilaa[4]**

[1]Department of Mathematics, Faculty of Science, Northern Border University, Kingdom of Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Northern Border University, Kingdom of Saudi Arabia
[3]Department of Mathematics, Faculty of Science, Northern Border University, Kingdom of Saudi Arabia
[4]Department of Electrical Engineering, Faculty of Engineering, Northern Border University, Kingdom of Saudi Arabia

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***Corresponding author:** Alaa Abalgaduir
Department of Mathematics, Faculty of Science and Art, Northern Border University, Kingdom of Saudi Arabia

**Abstract** **Original Research Article**

The differential equations, usually the goal is to find a solution. In other words, we want to find a function (or functions) that satisfies the differential equation. The technique we use to find these solutions varies, depending on the form of the differential equation with which we are working. Second-order differential equations have several important characteristics that can help us determine which solution method to use. In this section, we examine some of these characteristics and the associated terminology. An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (*DE*).
**Keywords:** differential equations, terminology, independent variables.


### 0.1.1 CLASSIFICATION BY TYPE

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an ordinary differential equation (*ODE*)
For example,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0 \tag{0.1.1}$$

**Partial differential equation PDE**

An equation involving partial derivatives of one or more dependent variables of two or more independent variables [4]. For example,

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{0.1.2}$$

### 0.1.2 CLASSIFICATION BY ORDER

The order of a differential equation (either *ODE* or *PDE*) is the order of the highest derivative in the equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0 \tag{0.1.3}$$

is a second-order ordinary differential equation.

### 0.1.3 CLASSIFICATION BY LINEARITY

let

$$F\,(x, y, y^{\cdot}, \ldots, y^{(n)}) = 0 \tag{0.1.4}$$

Nth-order ordinary differential equation in one dependent variable is said to be linear if F is linear in

$$y^{\cdot}, y^{\cdot\cdot}, \ldots, y^{(n)} \tag{0.1.5}$$

for example,

$$\frac{d^2y}{dx^2} + \sin y = 0 \quad (0.1.6)$$

is nonlinear second-order ordinary differential equation. And

$$\frac{d^2y}{dx^2} + \sin x = 0 \quad (0.1.7)$$

is linear second-order ordinary differential equation.

**0.1.4 Solution of an ODE**

Any function $\psi$, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a solution of the equation on the interval.

**0.1.5 Implicit Solution of an ODE**

A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation

$$F(x, y, y', \ldots, y^{(n)}) = 0 \quad (0.1.8)$$

On an interval I, provided that there exists at least one function $\psi$ that satisfies the relation as well as the differential equation on I.

**0.1.6 SYSTEMS OF DIFFERENTIAL EQUATIONS**

A system of ordinary differential equations is two or more equations involving the derivatives of two or more unknown functions of a single independent variable[**?**]. then a system of two first-order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y) \quad (0.1.9)$$

$$\frac{dy}{dt} = f(t, x, y) \quad (0.1.10)$$

A solution of a system such as

$$\frac{dx}{dt} = f(t, x, y) \quad (0.1.11)$$

$$\frac{dy}{dt} = f(t, x, y) \quad (0.1.12)$$

is a pair of differentiable functions $x = \psi_1(t)$ and $y = \psi_2(t)$.

**0.1.7 Two form of Second-Order Linear Differential Equations**

first form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \quad (0.1.13)$$

Where $P$, $Q$, $R$ and $G$ are continuous functions. if $G(x)=0$ for all $x$ Such equations are called homogeneous linear equations. we will get

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \quad (0.1.14)$$

If $G(x) /= 0$for all $x$ Such equations are called nonhomogeneous.

**0.1.8 Two type to solve homogeneous linear equations**

If we know two solutions $y_1$ and $y_2$ of such an equation, then the linear combination.

$$y = c_1y_1 + c_2y_2 \tag{0.1.15}$$

If $y_1$and $y_2$ and are linearly independent solutions of Equation

$$P\,(x)\frac{d^2y}{dx^2} + Q(x)\,\frac{dy}{dx}+R(x)y=0 \tag{0.1.16}$$

and $P\,(x)$ is never 0, then the general solution is given by

$$y(x) = c_1y(x)_1 + c_2y(x)_2 \tag{0.1.17}$$

Where $c_1$ and $c_2$ are arbitrary constants.

**0.1.9 Second form of Second-Order Linear Differential Equations**

$$\alpha\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy=0 \tag{0.1.18}$$

Where $a$, $b$ and $c$ are constants and $a$

0. let

$$y = e^{rx} \tag{0.1.19}$$

and

$$y^{\cdot} = re^{rx} \tag{0.1.20}$$

further more

$$y^{\cdot\cdot} = r^2e^{rx} \tag{0.1.21}$$

the solution of

$$\alpha\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy=0 \tag{0.1.22}$$

in this form

$$ar^2 + br + c = 0 \tag{0.1.23}$$

Called the auxiliary equation (or characteristic equation) [4].
the roots of the auxiliary equation.

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$

**0.1.10 CASE I**

$$b^2 - 4ac > 0 \tag{0.1.26}$$

If the roots $r_1$ and $r_2$ of the auxiliary equation are real and unequal, then the general solution of

$$\alpha\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy=0 \tag{0.1.27}$$

is

$$y = c_1e^r1x + c_2xe^r2x \tag{0.1.28}$$

**0.1.11 CASE 2**

$$b^2 - 4ac = 0 \tag{0.1.29}$$

if the auxiliary equation has only one real root, then the general solution of

$$\alpha\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy=0 \tag{0.1.30}$$

$$y = c_1e^{rx} + c_2xe^rx \tag{0.1.31}$$

### 0.1.12 CASE 3

$$b^2 - 4ac < 0 \quad (0.1.32)$$

The roots $r_1$ and $r_2$ of the auxiliary equation are complex numbers.

$$r_1 = \alpha + i\beta \quad (0.1.33)$$

and

$$r_2 = \alpha - i\beta \quad (0.1.34)$$

then the general solution of

$$\alpha \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad (0.1.35)$$

if

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \quad (0.1.36)$$

## 0.2 HOMOGENEOUS LINEAR FIRST ORDER SYS- TEMS WITH CONSTANT COEFFICIENTS

The general homogeneous linear first-order system

$$X^{\cdot} = AX \quad (0.2.1)$$

Where A is an $2 \times 2$ matrix of constants [3]. find to solution in the form

$$X = Ke^{\lambda t} \quad (0.2.2)$$

### 0.2.1 EIGENVALUES AND EIGENVECTORS

if

$$X = Ke^{\lambda t} \quad (0.2.3)$$

to be a solution vector of

$$X^{\cdot} = AX \quad (0.2.4)$$

then

$$X^{\cdot} = K\lambda e^{\lambda t} \quad (0.2.5)$$

so that the system becomes

$$K\lambda e^{\lambda t} = AKe^{\lambda t} \quad (0.2.6)$$

let

$$AK = \lambda K \quad (0.2.7)$$

or

$$(A - \lambda I)\, K = 0 \quad (0.2.8)$$

Thus, to find a nontrivial solution $X$ of

$$X^{\cdot} = AX \quad (0.2.9)$$

find a nontrivial vector $K$ that satisfies

$$(A - \lambda I)\, K = 0 \quad (0.2.10)$$

the equation

$$det\,(A - \lambda I) = 0 \quad (0.2.11)$$

is called the characteristic equation of the matrix $A$. hence

$$X = Ke^{\lambda t} \quad (0.2.12)$$

will be a solution of the system differential equation $\Longleftrightarrow$ $\lambda$ is an eigen value of $A$ and $K$ is an eigen vector corresponding to $\lambda$.

**Three cases of eigen value**

**Case 1**

The 2 × 2 matrix A possesses n distinct real eigenvalues

$$\lambda_1, \lambda_2 \tag{0.2.13}$$

then a set of n linearly independent eigenvectors

$$K_1, K_2 \tag{0.2.14}$$

then

$$X_1 = K_1 e^{\lambda} 1t \tag{0.2.15}$$

and

$$X_2 = K_2 e^{\lambda} 2t \tag{0.2.16}$$

is a fundamental set of solutions of

$$X^{\cdot} = AX \tag{0.2.17}$$

then

$$X = K_1 e^{\lambda} 1t + K_2 e^{\lambda} 2t \tag{0.2.18}$$

**Case 2**

if $(\lambda - \lambda_1)^2$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^3$ is not a factor, then$\lambda_1$ is said to be an eigenvalue of multiplicity 2.

For some 2×2matrices A it may be possible to find m linearly independent eigenvectors $K_1$, $K_2$. corresponding to an eigenvalue $\lambda_1$ of multiplicity 2. then the general solution is

$$X = c_1 K_1 e^{\lambda} 1t + c_2 K_2 e^{\lambda} 2t \tag{0.2.19}$$

If there is only one eigenvector corresponding to the eigenvalue $\lambda_1$ of multiplicity 2, then 2 linearly independent solutions of the form

$$X_1 = K_1 1 e^{\lambda} 1t \tag{0.2.20}$$

$$X_2 = K_2 1 t e^{\lambda} 2t + K_2 2 e^{\lambda} 1t \tag{0.2.21}$$

Where $Kij$ are column vectors.

**Case 3**

Let $K_1$ be an eigenvector of the coefficient matrix A (with real entries) corresponding to the complex eigenvalue

$$\lambda_1 = \alpha + \beta\iota \tag{0.2.22}$$

then the solution are

$$X_1 = (B_1 \cos \beta t - B_2 \sin \beta t) e^{\alpha} t \tag{0.2.25}$$

$$X_2 = (B_2 \cos \beta t - B_1 \sin \beta t) e^{\alpha} t \tag{0.2.26}$$

are linearly independent solutions.

B1 and B2 denote the column vectors.

## RESULT AND DISCUSSION

We can solve second-order, linear, homogeneous differential equations with constant coefficients by finding the roots of the associated characteristic equation. The form of the general solution varies, depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots. The three cases are summarized

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