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Exploring Wave Dynamics: Symmetry Reductions and Similarity Solutions to the (2+1) Dimensional ZK-BBM Equation

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Abstract **Review Article**

This study uses the long-established Lie symmetry approach to work out the Zakharov-Kuznetsov- Benjamin-Bona-Mahony (ZK-BBM) equation, a nonlinear partial differential equation (PDE) widely applied in modelling wave propagation across numerous materials. By distinguishing symmetries and group-invariant solutions, the ZK-BBM equation can be curtailed, and a new exact solution can be deduced. The work also probes similarity solutions to the equation, exhibiting various examples and their outcomes for wave dynamics. Eventually, conservation laws related to the ZK-BBM equation are evaluated using adjoint equations and their symmetries. The behaviour of the ZK-BBM equation and its solutions gives profound knowledge that ushers in the importance of this comprehensive study, which greatly impacts numerous study fields like quantum physics, fluid dynamics, and elasticity theory.
Keywords: Lie symmetries, ZK-BBM equation, group-invariant solutions, similarity solutions, conservation laws, partial differential equations, wave propagation, and nonlinear dynamics.

1. INTRODUCTION

The symmetries [1, 2] do play a vital role in the investigation of integrable systems in the study of partial differential equations (PDEs) which exhibit intricacy. The convoluted systems, which are designed by their infinite symmetries, can be competently investigated using symmetry group approaches [3–5]. A precise solution for complex PDEs can be attained by these key methods. A few techniques to calculate the Lie point symmetries of a nonlinear equation [1, 6] have emerged because of their effectiveness. Nonlinear PDEs are in several scientific areas, including condensed matter physics, fluid mechanics, plasma physics, and optics. Finding exact solutions to PDEs is a key issue in mathematics and physics.

The classical Lie group technique [7, 8] is a traditional practice for studying differential equations using continuous transformation groups. Literal solutions of several PDEs can be attained for travelling wave solutions, similarity solutions, soliton wave solutions, and fundamental solutions by using the Lie group approach. Compared with the non-classical Lie group approach, it expands the classical technique by considering additional restrictions which would be uniform under the action of the symmetry group. Clarkson-Kruskal (CK) presented the direct symmetry method [9, 10] which is a relatively easy approach that, as result, reduces the process of finding symmetry by abolishing the lengthy computations involved with traditional approaches. This study technique focuses on identifying reductions from PDEs to ordinary differential equations (ODEs), which can be easily solved. There is another important technique, which is the compatibility method [11, 12] and investigates the compatibility conditions of overdetermined differential equation systems. This beneficial method is being used to find hidden symmetries which are hard to find using traditional methods. The study of symmetries in perturbed partial differential equations [13] requires an equal balance between understanding the stability of existing symmetries and identifying new ones. The importance of this study is that it is not just for theoretical investigations but also being used for such applications, which are real-world examples like elasticity theory, fluid dynamics, and quantum mechanics, to understand the system and how it will perform with a slight perturbation. The generalised symmetry approach finds the transformations, which are not always point transformations; hence, developing these methods lead to more suitable approaches. A specific solution that cannot be determined by using the conventional symmetry approach can be attained by using the conditional symmetry approach, which is another noteworthy approach. This study is also extended to find the lie symmetries, conservation laws and exact solutions of fractional-order partial differential equations [14] and the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi Hermite polynomials [15].

This study aims to apply the classical Lie symmetry method to a specific nonlinear partial differential equation known as the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation, which is given by:

$$u_t + u_x + puu_x + qu_{xxt} + qu_{yyx} = 0. \ldots\ldots\ldots (1)$$

The existence of mixed partial derivatives indicates that they are an extended form of the standard Korteweg-De Vries (KdV) equation, containing both two-dimensional effects and higher order dispersion factors. The coefficients p and q in the equation are real constants that characterise the amplitude and dispersion properties of the wave, respectively.

To find the solutions for the ZK-BBM equation, Wazwaz [16, 17] implied the sine-cosine approach and the extended tanh method on solitons, periodic solutions, and complex solutions. The f-expansion method was used for improved analysis and generation of exact results by Abdou [18]. This approach requires expanding the solutions, using a finite set of functions can provide a systematic method for discovering new and potentially more general solutions. The bifurcation method of a dynamical system was used by Song and Yang [19] to address the system problem. It is important to understand the qualitative behaviour of solutions, particularly for identifying travelling wave solutions and figuring out their stability and bifurcation features.

This study is organized in such a way that Section 2 discusses the findings of symmetries, Lie symmetry groups, and invariant solutions for equation (1). In Section 3, we find the best reductions of the equation (1), while Section 4 is dedicated to the derivation of new exact solutions. Section 5 contains the conservation laws associated with equation (1). This study ends with Section 6, which concludes our work.

2. Symmetries and Lie symmetry groups

The fundamental goal of using the Lie symmetry approach in this context is to find symmetries of a given PDE, with another goal of determining exact solutions. For equation (1), the corresponding vector field can be expressed as follows:

$$\mathbf{X} = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}, \ldots\ldots\ldots\ldots (2)$$

Which is the infinitesimal generator of the symmetry group. The equation (1) contains the highest order of three, so we define its third prolongation, which expands this generator to include derivatives up to the third order. This is essential for collecting the behaviour of solutions under transformations involving second-order derivatives present in equation (1). The third prolongation is written in the following form:

$$\mathbf{X}^{(3)} = \mathbf{X} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xxt}\frac{\partial}{\partial u_{xxt}} + \zeta_{yyx}\frac{\partial}{\partial u_{yyx}}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where the functions $\zeta_t, \zeta_x, \zeta_{xxt}, \zeta_{yyx}$ are defined in terms of $\xi^1, \xi^2, \xi^3, \eta$, and the derivatives of ζ, ensuring full understanding about the behaviour of the PDE's under symmetry transformations. From $\mathbf{X}^{(3)}(F)\big|_{F=0} = 0$, which means applying third prolongation to the equation (1), it follows as:

$$\zeta_t + \zeta_x + pu_x\zeta + pu\zeta_x + q\zeta_{xxt} + q\zeta_{yyx} = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

By setting the coefficient of the polynomial in equation (4) to zero, a system of differential equations emerges, determining the forms of the functions ξ, ξ^2, ξ^3, and η. Solving this system reveals the symmetry parameters:

$$\xi^1 = e_3,\ \xi^2 = \frac{1}{2}e_1 y + e_4,\ \xi^3 = e_1 t + e_2,\ \eta = -e_1\frac{pu+1}{p}, \ldots\ldots\ldots\ldots\ldots (5)$$

And the arbitrary constants e_1, e_2, e_3, and e_4 are important in construction of the corresponding symmetries and generating exact solutions.

The resulting symmetries can be expressed as follows:

$$\mathcal{X} = e_3 u_x + \left(\frac{1}{2}e_1 y + e_4\right)u_y + (e_1 t + e_2)u_t - e_1\frac{pu+1}{p}u, \ldots\ldots\ldots\ldots\ldots (6)$$

Representing the transformations under which the PDE maintains its form. Each term in equation (6) corresponds to a specific transformation in the (x, y, t, u) space, indicating how the solution u changes under the action of the symmetry.

The vector field is defined as:

$$\mathbf{X} = e_3 \frac{\partial}{\partial x} + \left(\frac{1}{2} e_1 y + e_4\right) \frac{\partial}{\partial y} + (e_1 t + e_2) \frac{\partial}{\partial t} - e_1 \frac{pu+1}{p} \frac{\partial}{\partial u}. \quad \ldots\ldots\ldots\ldots\ldots.. (7)$$

To obtain exact solutions from those already known for equation (1), we have to identify the related Lie symmetry groups and to do this, we use the initial problems. The equation takes the form of an initial value problem, where the dependent variables (x, y, t, u) are transformed into new variables $(\bar{x}, \bar{y}, \bar{t}, \bar{u})$ under the action of a parameterized symmetry group. The parameter Varepsilon serves as a continuous scaling factor, allowing for infinitesimal transformations.

The differential equation:

$$\frac{d}{d\varepsilon}(\bar{x}, \bar{y}, \bar{t}, \bar{u}) = (x, y, t, u), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (8)$$

Specifies how the transformed variables change concerning the parameter ε. This equation essentially describes the infinitesimal action of the Lie symmetry group on the variables (x, y, t, u). By integrating this differential equation with respect to ε, we obtain the transformed variables (x, y, t, u) as functions of $(\bar{x}, \bar{y}, \bar{t}, \bar{u})$. The initial conditions are:

$$(\bar{x}, \bar{y}, \bar{t}, \bar{u})|_{\varepsilon=0} = (x, y, t, u), \quad \ldots\ldots\ldots\ldots\ldots.. (9)$$

Define the values of the transformed variables at the starting point $(\varepsilon = 0)$.

From equation (8), we can derive the Lie symmetry group denoted as $\mathbf{g}: (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \rightarrow (x, y, t, u)$. This group represents the transformations that preserve the form of equation (1). By analysing different functions ξ^1, ξ^2, ξ^3, and η appearing in the expression for the symmetry $\mathcal{X}$, we can solve equation (8) to obtain particular Lie symmetry groups. These groups, denoted as $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and g_4, are described as follows:

$$\begin{aligned}
&\mathbf{g}_1: (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \longrightarrow (x + \varepsilon, y, t, u), \\
&\mathbf{g}_2: (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \longrightarrow (x, y + \varepsilon, t, u), \\
&\mathbf{g}_3: (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \longrightarrow (x, y, t + \varepsilon, u), \\
&\mathbf{g}_4: (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \longrightarrow \left(x, y + e^{\frac{1}{2}\varepsilon}, t + e^{\varepsilon}, u + e^{-\varepsilon} - \frac{1}{p}\right).
\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots.. (10)$$

These Lie symmetry groups represent translations along the x, y and t axes, as well as a more complex transformation involving exponential terms. By applying these symmetry groups $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and $\mathbf{g}_4$ to known solutions of equation (1), represented by the function v new exact solutions can be obtained:

$$\begin{aligned}
&\mathbf{u}_1 = v(x - \varepsilon, y, t), \\
&\mathbf{u}_2 = v(x, y - \varepsilon, t), \\
&\mathbf{u}_3 = v(x, y, t - \varepsilon), \\
&\mathbf{u}_4 = v\left(x, y - e^{-\frac{1}{2}\varepsilon}, t - e^{-\varepsilon}\right) - \frac{1}{p} + e^{-\varepsilon}.
\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.... (11)$$

For instance, deal with the solution, which is a periodic wave solution [16] of the equation (1)

$$u(x, y, t) = \frac{3(d-1)}{2p} \sec^2\left(\frac{1}{2\sqrt{q}} \xi^1\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (12)$$

Where $q > 0$ and $\xi^1 = x + y - dt$.

By applying the $\mathbf{u}_4$, a new exact solution of equation (1) can be derived, involving the exponential term $e^{-\varepsilon}$ in the argument of the periodic wave function.

$$u(x, y, t) = \frac{(3dc-1)}{2a} \sec^2\left(\frac{1}{2\sqrt{q}} \xi^1\right) - \frac{1}{p} + e^{-\varepsilon}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13)$$

Where $q > 0$ and $\xi^1 = x + y - e^{-\frac{1}{2}\varepsilon} - d(t - e^{-\varepsilon})$.

3. Symmetry Reduction of ZK-BBM Equation

In this section, we reduce the equation (1) with the help of the symmetries (5). To do this, we need to discuss some cases:

Case 3.1:

We consider the conditions where $e_1 \neq 0, e_2 = 0, e_3 = 0, e_4 = 0$ and substituting these values into the expression for the symmetry $\mathcal{X}$, we will have:

$$\mathcal{X} = \frac{1}{2} e_1 y u_y + e_1 t u_t - e_1 \frac{au+1}{p} u, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (14)$$

Subsequently solving the differential equation $\mathcal{X} = 0$, yields specific expressions for the variables ξ^1, η and u as shown in equation (15).

$$\xi^1 = x,\ \eta = \frac{y^2}{t},\ u = \frac{1}{t}v(\xi^1, \eta) - \frac{1}{a}. \quad \text{(15)}$$

Using these derived expressions in equation (1) leads to the reduced form given as follows:

$$-v - \eta v_\eta + v v_{\xi^1} + b\left(-v_{\xi^1\xi^1} - \eta v_{\xi^1\xi^1\eta} + 2v_{\eta\xi^1} + 4\eta v_{\eta\eta\xi^1}\right) = 0. \quad \text{(16)}$$

Case 3.2:

We explore this case when $e_1 = 0, e_2 \neq 0, e_3 \neq 0, e_4 \neq 0$, and substituting these conditions into the expression for $\mathcal{X}$, we get:

$$\mathcal{X} = e_3 u_x + e_4 u_y + e_2 u_t. \quad \text{(17)}$$

Solving for $\mathcal{X} = 0$, we obtain the expressions for ξ^1, η and u.

$$\xi^1 = x - \frac{e_3}{e_2}t,\ \eta = y - \frac{e_4}{e_2}t,\ u = v(\xi^1, \eta). \quad \text{(18)}$$

By using these expressions (18) into equation (1) results in the reduced form:

$$\left(1 - \frac{e_3}{e_2}\right)v_{\xi^1} - \frac{e_4}{e_2}v_\eta + p v v_{\xi^1} + q\left(-\frac{e_3}{e_2}v_{\xi^1\xi^1\xi^1} - \frac{e_4}{e_2}v_{\xi^1\xi^1\eta} + v_{\eta\eta\xi^1}\right) = 0. \quad \text{(19)}$$

Case 3.3:

We consider the conditions where $e_1 = 0, e_2 \neq 0, e_3 = 0, e_4 \neq 0$. Substituting these values into the expression for $\mathcal{X}$, we obtain the following form:

$$\mathcal{X} = e_4 u_y + e_2 u_{t.} \quad \text{(20)}$$

Solving for $\mathcal{X} = 0$, we obtain expressions for ξ^1, η and u.

$$\xi^1 = x,\ \eta = y - \frac{e_4}{e_2}t,\ u = v(\xi^1, \eta). \quad \text{(21)}$$

Substituting these expressions (21) into equation (1) yields the reduced form, represented as follows:

$$v_{\xi^1} - \frac{e_4}{e_2}v_\eta + q v v_{\xi^1} + q\left(-\frac{e_4}{e_2}v_{\xi^1\xi^1\eta} + v_{\eta\eta\xi^1}\right) = 0. \quad \text{(22)}$$

4. Similarity solutions of ZK-BBM equation

We explore various cases to obtain new solutions of equation (1) by dealing with the reduced equations (16), (19), and (22). Those cases are studied in the following ways:

Case 4.1:

Here, we say that equation (16) admits a solution that looks like the following:

$$v = \Phi(\eta)\xi^1 + \Psi(\eta), \quad \text{(23)}$$

Where $\Phi(\eta)$ and $\Psi(\eta)$ are unknown functions that we have to determine. Using the equation (23) inside the equation (16) gives the following:

$$v = \frac{1}{1 + e_1\eta}\xi^1 + \Psi(\eta), \quad \text{(24)}$$

Then we find a new exact solution of equation (1) that looks like the following:

$$\mathbf{u_1} = \frac{1}{p(t + c_1^* y^2)}\left(x + 2qc_1^* \ln\left(t + c_1^* y^2\right) - 4qc_1^* \ln\ y + c_2^*\right) - \frac{8qtc_1^*}{p\left(t + c_1^* y^2\right)^2} - \frac{1}{p}, \quad \text{(25)}$$

Which offers ideas about the behaviour of the equation (1) under specific conditions determined by the constants c_1^* and c_2^*.

Case 4.2:

We study equation (19) by employing the G'/G-expansion method [20] and looking for the solutions, which are travelling wave solutions.

$$v = \sum_{i=1}^{N} \lambda_i \left(\frac{G'(\Psi)}{G(\Psi)}\right)^i + \lambda_0,\ \lambda_j \neq 0, \quad \text{(26)}$$

Where λ_i are constants that we find later and $\Psi = k\xi^1 + l\eta$. It is observed that $j = 2$ by balancing v_{ξ^1} and vv_{ξ^1} in equation (17). Let the solutions of equation (17) be of the form:

$v = \lambda_2 \left(\frac{G'(\Psi)}{G(\Psi)}\right)^2 + \lambda_1 \left(\frac{G'(\Psi)}{G(\Psi)}\right) + \lambda_0,\ \lambda_2 \neq 0,$ ………..…….. (27)

with $G(\Psi)$ satisfying the second-order linear ODE
$G''(\Psi) + \kappa G'(\Psi) + \gamma G(\Psi) = 0,$ ……………………….. (28)

Where the constants to be determined later are $\lambda_0, \lambda_1, \lambda_2, \kappa, \gamma$ and $\Psi = k\xi^1 + l\eta$. Substituting equation (27) into equation (19) along with equation (28) and fixing the values of the coefficients of $(G'(\Psi)/G(\Psi))^i, (i = 0, \ldots, 4)$ to zero results in a system of equations concerning $\lambda_0, \lambda_1, \lambda_2, \Upsilon$, and l.

After a detailed derivation process, we obtain three different types of travelling wave solutions and for every solution, we can have further three cases on $\lambda^2 - 4\gamma$ which are given below:

Case 4.2.1:
When $\lambda^2 - 4\gamma > 0$,

$$u_2 = \frac{3q}{pe_2}(\lambda^2 - 4\gamma)\left(e_3 k + e_4 l - e_2 \frac{l^2}{k}\right)\frac{c_1^* \sin \Upsilon\Psi + c_2^* \cos \Upsilon\Psi}{c_1^* \cos \Upsilon\Psi + c_2^* \sin \Upsilon\Psi} + \frac{2q}{pe_2}(\lambda^2 - 4\gamma)\left(-e_3 k - e_4 l + e_2 \frac{l^2}{k}\right) + \frac{1}{pe_2}\left(e_4 \frac{l}{k} - e_2 + e_3\right). \quad \text{…………….. (29)}$$

Case 4.2.2:
When $\lambda^2 - 4\gamma < 0$,

$$u_3 = -\frac{3q}{pe_2}(4\gamma - \lambda^2)\left(e_3 k + e_4 l - e_2 \frac{l^2}{k}\right)\frac{c_1^* \sin \Upsilon\Psi + c_2^* \cos \Upsilon\Psi}{c_1^* \cos \Upsilon\Psi + c_2^* \sin \Upsilon\Psi} + \frac{2q}{pe_2}(4\gamma - \lambda^2)\left(-e_3 k - e_4 l + e_2 \frac{l^2}{k}\right) + \frac{1}{pe_2}\left(e_4 \frac{l}{k} - e_2 + e_3\right). \quad \text{…………….. (30)}$$

Case 4.2.3:
When $\lambda^2 - 4\gamma = 0$,

$$u_4 = \frac{12q}{pe_2}\left(e_3 k + e_4 l - e_2 \frac{l^2}{k}\right)\frac{c_1^* \sin \Upsilon\Psi + c_2^* \cos \Upsilon\Psi}{c_1^* \cos \Upsilon\Psi + c_2^* \sin \Upsilon\Psi} + \frac{1}{pe_2}\left(e_4 \frac{l}{k} - e_2 + e_3\right), \quad \text{………. (31)}$$

where $\Psi = kx + ly - \left(\frac{ke_3 + ld - 4}{e_2}\right)t$.

Case 4.3:
Let $v = v(\Psi)$, where $\Psi = k\xi^1 + l\eta$ and equation (22) becomes an ODE:

$$\left(k - \frac{e_4}{e_2} l\right) v' + pkvv' + q\left[-\frac{e_4}{e_2} k^2 l + kl^2\right] v''' = 0, \quad \text{……………………….. (32)}$$

here, we integrate two times with respect to Ψ in equation (32), obtain:

$$(v')^2 = \Gamma_1 v^3 + \Gamma_2 v^2 + \Gamma_3 v, \quad \text{……………………………………………….. (33)}$$

Where $\Gamma_1 = \frac{e_2 p}{3ql}(e_4 k - e_2 l), \Gamma_2 = \frac{e_2 k - e_4 l}{qkl}(e_4 k - e_2 l)$, and Γ_3 is a constant.

Given that solutions to equation (33) have been provided in [21], we can derive several similarity solutions for equation (1) as outlined below:

Case 4.3.1:
When $\Gamma_1 = 4\Theta^2, \Gamma_2 = 4(-\Theta^2 - 1), \Gamma_3 = 4$,

$$u_5 = s\vartheta^2\left[kx + ly - \frac{e_4}{e_2} lt\right]. \quad \text{…………….. (34)}$$

Case 4.3.2:
When $\Gamma_1 = 4\Theta^2, \Gamma_2 = 4(-2\Theta^2 + 1), \Gamma_3 = 4(1 - \Theta^2)$,

$$u_6 = d\vartheta^2\left[kx + ly - \frac{e_4}{e_2} lt\right]. \quad \text{…………….. (35)}$$

Case 4.3.3:
When $\Gamma_1 = -4, \Gamma_2 = -4(\Theta^2 - 2), \Gamma_3 = 4(\Theta^2 - 1)$,

$$u_7 = f\vartheta^2\left[kx + ly - \frac{e_4}{e_2} lt\right]. \quad \text{………………………….. (36)}$$

Case 4.3.4:
When $\Gamma_1 v < 0, \Gamma_2 > 0, \Gamma_3 = 0$

$$u_8 = -\frac{\Pi_2}{\Pi_1}\operatorname{sech}^2\left[\frac{\sqrt{\Pi_2}}{2}\left(kx + ly - \frac{e_4}{e_2}lt\right)\right]. \qquad (37)$$

Case 4.3.5:
When $\Gamma_1 v > 0, \Gamma_2 > 0, \Gamma_3 = 0$

$$u_9 = \frac{\Pi_2}{\Pi_1}\operatorname{csch}^2\left[\frac{\sqrt{\Pi_2}}{2}\left(kx + ly - \frac{e_4}{e_2}lt\right)\right]. \qquad (38)$$

Case 4.3.6:
When $\Gamma_2 < 0, \Gamma_3 = 0$

$$u_{10} = \frac{\Pi_2}{\Pi_1}\sec^2\left[\frac{\sqrt{\Pi_2}}{2}\left(kx + ly - \frac{e_4}{e_2}lt\right)\right]. \qquad (39)$$

The equation (33) leads us to a variety of similarity solutions from equation (34) to equation (39), each characterized by different combinations of constants $\Theta, \vartheta, e_1, e_2, e_3$ and e_4.

5. Conservation laws of ZK-BBM equation

The conservation laws [1, 22] of the ZK-BBM equation are obtained by using the adjoint equation and its symmetries. For equation (1), the adjoint equation takes the form:

$$z_t + z_x + puz_x + qz_{xxt} + qz_{yyx} = 0, \qquad (40)$$

and the Lagrangian associated with this equation is:

$$\mathcal{L} = z\left(u_t + u_x + puu_x + qu_{xx}z_t + qu_{yy}z_x\right). \qquad (41)$$

To derive the conservation laws, the Lie point, Lie-Backlund, and non-local symmetries of (1) and its adjoint equation [23] are very important. The components of the conservation vector $(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3)$ are defined as follows:

$$\begin{aligned}\mathrm{T}^i &= \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right)\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right)\right] + D_jD_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\ (i = 1,2,3),\end{aligned} \qquad (42)$$

Where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is a Lie characteristic function.
The conserved vector related to an operator $\mathcal{V}$ given by:

$$\mathcal{V} = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}. \qquad (43)$$

The conservation law equation is represented in the following form:

$$D_t(\mathrm{T}^1) + D_x(\mathrm{T}^2) + D_y(\mathrm{T}^3) = 0, \qquad (44)$$

Where the conserved vector $\mathrm{T} = (\mathrm{T}^1, \mathrm{T}^2, \mathrm{T}^3)$ is given by the expression in equation (42). The components of this conservation vector are further defined as follows:

$$\begin{aligned}\mathrm{T}^1 &= w^1[z(1+pu) - D_x(qzz_t)] + ku_{yy}zw^2 + qzz_tD_x(w^1), \\ \mathrm{T}^2 &= (\xi^1)^2L - D_y(kzz_x)w^1 + qzz_xD_y(w^1), \\ \mathrm{T}^3 &= (\xi^1)^3L + w^1 + qu_{xx}zw^2.\end{aligned} \qquad (45)$$

The system of equations (45) defines the conservation law of equation (1) and equation (40), associated with any operator z accepted by equation (1).

We investigate a more detailed computation for a specific operator z defined as:

$$z = \frac{1}{2}y\,\partial y + t\,\partial t - \frac{pu+1}{p}\partial u. \qquad (46)$$

The expressions for w^1 and w^2 are given by:

$$w^1 = -\frac{pu+1}{p} - \frac{1}{2}yu_y - tu_t, \qquad (47)$$

$$w^2 = \frac{1}{2}z - \frac{1}{2}yz_y - tz_t. \qquad (48)$$

Substituting these into equation (42) and evaluating for the sixth-order Lagrangian equation (41), we obtain the components of the conservation vector as described as:

$$\begin{aligned}
T^1 &= -\left(u+\frac{1}{p}+\frac{1}{2}yu_y+tu_t\right)(z+puz-qz_xz_t-qzz_{tx})\\
&+qu_{yy}z\left(\frac{1}{2}z-\frac{1}{2}yz_y-tz_t\right)-qzz_t\left(u_x+\frac{1}{2}yu_{yx}+tu_{tx}\right),\\
T^2 &= \frac{1}{2}yz(u_t+u_x+puu_x+qu_{xx}z_t+qu_{yy}z_x)-q(z_yz_x+zz_{xy})\\
&\cdot\left(u+\frac{1}{a}+\frac{1}{2}yu_y+tu_t\right)-qzz_x\left(u_y+\frac{1}{2}u_y+\frac{1}{2}yu_{yy}+tu_{ty}\right),\\
T^3 &= tz(u_t+u_x+puu_x+qu_{xx}z_t+qu_{yy}z_x)\\
&-\left(u+\frac{1}{p}+\frac{1}{2}yu_y+tu_t\right)+qu_{xx}z\left(\frac{1}{2}z-\frac{1}{2}yz_y-tz_t\right).
\end{aligned} \quad \text{……….. (49)}$$

So, these conserved vectors satisfy the equation (1).

6. CONCLUSION

Finally, this work studied the classical Lie symmetry approach as applied to the (ZK-BBM) equation, providing insight on its symmetries, reductions, exact solutions, and conservation laws. We determined the ZK-BBM equation's Lie point symmetries using the classical Lie symmetry approach, exposing its fundamental invariant characteristics. These symmetries were identified by certain vector fields, which made it possible for us to solve the problem exactly. The ZK-BBM equation's simpler versions were produced via reductions, making it easier to look at the dynamics of the equation under various circumstances. Further, we explored similarity solutions to the ZKBBM equation, leveraging the reduced equations to derive explicit solutions. Through various cases and mathematical techniques such as the G'/G-expansion m differential equations, we obtained a range of new explicit solutions, each revealing distinct aspects of the equation's behaviour. Finally, we investigated the conservation laws associated with the ZK-BBM equation, utilising the adjoint equation and its symmetries. By examining the Lagrangian and utilising Noether's theorem, we uncovered conserved quantities that provide valuable insights into the system's dynamics and energy conservation properties.

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Data Availability

The datasets used in this study are not publicly available due to confidentiality reasons but can be requested from the corresponding author. Participants only consented to the publication of aggregated data.

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