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Regular Triangling a Circle with Straightedge and Compass in Euclidean Geometry

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Abstract **Review Article**

No scientific theory lasts forever, but specific research and discoveries continuously build upon each other. The three classic ancient Greek mathematical challenges likely referring to are **"Doubling the Circle"**, "Trisecting An Angle" & "Squaring The Circle", all famously proven Impossible under strict compass-and-straightedge constraints, by Pierre Wantzel (1837) using field theory and algebraic methods, then also by Ferdinand von Lindemann (1882) after proving π is transcendental. These original Greek challenges remain impossible under classical rules since their proofs rely on deep algebraic/transcendental properties settled in the 19th century. Recent claims may involve reinterpretations or unrelated advances but do overturn these conclusions above. Among these, the "**Squaring the Circle**" problem and related problems involving π have captivated both professional and amateur mathematicians for millennia. The title of this paper refers to the concept of "constructing a regular triangle that has the exact area of a given circle," or "Triangling the Circle". This research idea arose after the **"Squaring the Circle"** problem was studied and solved and published in "SJPMS" in 2024.This paper presents an exact solution to constructing a regular triangle that is concentric with and has the same area as any given circle. The solution does not rely on the number π and adheres strictly to the constraints of Euclidean geometry, using only a straightedge and compass. The technique of "ANALYSIS" is employed to solve the "Triangling the Circle" problem precisely and exactly with only a straightedge and compass, without altering any premise of the problem. This independent research demonstrates the solution to the challenge using only these tools. All mathematical tools and propositions in this solution are derived from Euclidean geometry. The methodology involves geometric methods to arrange the given circle and its equal-area regular triangle into a concentric position. Building on this method of exact "**Triangling A Circle**," one can deduce an equivalent problem to formulate a new mathematical challenge: "**Hexagoning A Circle**" (i.e., constructing a regular hexagon that has the same area as a given circle, using only a straightedge and compass).

Keywords: Triangling the Circle, Regular Triangling the Circle, Circle Triangulated, Triangle Area of a Circle, Constructing a Regular Triangle with the Same Area as a Circle, Euclidean Geometry, Straightedge and Compass.

INTRODUCTION

In the past, knowledge was often considered scientific if it could be confirmed through specific evidence or experiments. However, Karl Popper, in his book *Logik der Forschung* (The Logic of Scientific Discovery), published in 1934, demonstrated that a fundamental characteristic of scientific hypotheses is their ability to be proven wrong (falsifiability). Anything that cannot be refuted by evidence is temporarily regarded as true until new evidence emerges. For instance, in astronomy, the Big Bang theory is widely accepted, but in the future, anyone who discovers a flaw in this theory will be acknowledged by the entire physics community. Furthermore, no scientific theory lasts forever; rather, it is specific research and discoveries that continually build upon one another [1].

About three thousand years ago, three well-known problems in ancient Greek mathematics emerged. Among them, Hippocrates meticulously studied the problem of squaring the circle. The problem is stated as follows: Using only a straightedge and a compass, is it possible to construct a square with an area equal to a given circle? These problems were proven unsolvable using only straightedge and compass in the 19th century. In 1882, mathematician Ferdinand von Lindemann proved that π is an irrational number, which means that it is impossible to construct a square with the same area as a given circle using only a straightedge and a compass, as posed by Hippocrates. *One of the most fascinating aspects of this problem is that it has captured the interest of mathematicians throughout the history of mathematics. From the earliest mathematical documents to today's mathematics, the problem and its relation to π have intrigued both professional and amateur mathematicians for millennia.*

A significant step forward in proving the impossibility of squaring the circle occurred in 1761 when Lambert proved that π is irrational [2, 3]. This, however, was not sufficient to demonstrate the impossibility of squaring the circle using a straightedge and compass, as certain algebraic numbers can be constructed with these tools. Despite the proof of the impossibility of squaring the circle, the problem has continued to captivate mathematicians and the general public alike, remaining an important topic in the history and philosophy of mathematics.

In 1837, French mathematician L. Wantzel proved that the three classical problems of ancient Greece are impossible to solve using only a straightedge and compass. *Although based on algebra, Pierre Laurent Wantzel (1837) declared the verdict, "Arbitrary angle trisection is impossible", I experienced that my achieved procedure was possibly not arrested by this verdict [4]. After applying my method to trisect an arbitrary angle, I hope readers can trust algebra cannot affect the successful geometrical constructions of angle trisection.*

Therefore, the problems of squaring the circle, doubling the cube, and trisecting an angle have been studied for centuries and remain unsolvable with these tools to this day. As of December 2022, no mathematician has found exact solutions to classical problems such as "Doubling the Cube," "Squaring the Circle," or "Trisecting an Angle" using only a compass and straightedge. However, in 2023, my research, published in my paper on the "Squaring the Circle" problem, provides a counter-proof to the impossibility stated by Wantzel in 1837, [4].

Starting from accepted premises, without proof, one uses deductive reasoning to arrive at theorems and corollaries. With different premises, we develop different mathematical systems. For example, the premise "from a point outside a line, only one parallel line can be drawn to the given line" leads to Euclidean geometry. If we assume that no parallel lines can be drawn from that point, we enter the realm of Riemannian geometry. Alternatively, Lobachevskian geometry assumes that an infinite number of parallel lines can be drawn through that point.

It is important to clarify that this does not imply a square with an equal area to a circle does not exist. If the circle has an area A, a square with a side length equal to the square root of A would have the same area. It does not imply that it is impossible to solve the problem using only a straightedge and compass.

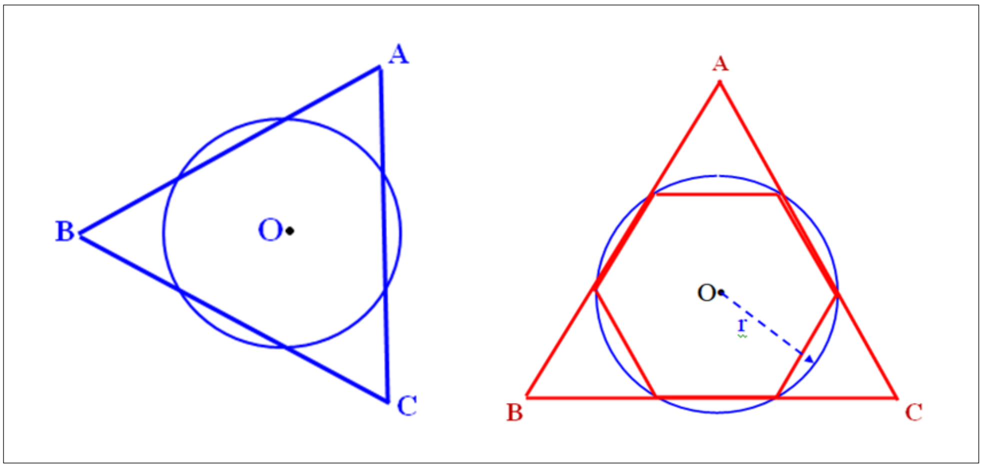
Thus, I have provided an exact solution to the "Squaring the Circle" problem without altering the premises of Euclidean geometry or the constraints of straightedge and compass. This solution was published in SJPMS [6]. In 2024, I also solved the inverse problem, titled "Circling the Square with Straightedge & Compass in Euclidean Geometry," which was published in [7].

"Regular Triangling A Circle" refers to the process of "*constructing a regular triangle that has the same area as a given circle and is concentric with that circle*." This concept came to me spontaneously after solving the problem of "squaring the circle." I thought, "*If one can square the circle, can one also regularly triangulate the circle using only a straightedge and compass in Euclidean Geometry?*" In other words, "Regular Triangling A Circle" is a new challenge problem that arose from the exact solution presented in my "Squaring the Circle" paper, published in *SJPMS* [6]. Therefore, the "Regular Triangling A Circle" challenge has not existed in the field of mathematics until it was solved and published recently.

In seeking solutions to such problems, geometers developed a special technique called "ANALYSIS." They would assume the problem had been solved, and by investigating the properties of the solution, they would work backward to identify an equivalent problem that could be solved based on the given conditions. To obtain the formally correct solution to the original problem, geometers would then reverse the process: starting with the data to solve the equivalent problem derived through analysis, and then using that solution to solve the original problem. This reversed procedure is known as "SYNTHESIS." I have adopted both the "ANALYSIS" and "SYNTHESIS" techniques to solve the "Regular Triangling A Circle" problem exactly, using only a straightedge, compass, in Euclidean Geometry, without involving the irrational number π.

The inspiration for this research arose after the solution to the "Squaring the Circle" problem was published in SJPMS in 2024. If it is possible to square a circle using Euclidean geometry, the question arises: why is it difficult to triangulate a circle? In this context, the given circle contains an inscribed hexagon with three non-consecutive sides extending from it, leading to an equilateral triangle. This equilateral triangle serves as the solution to the new problem of "Regular Triangling of a Circle." The remaining task is to prove that the area of this triangle is exactly equal to the area of the given circle. This paper also introduces a new mathematical tool, the "Triangling Ruler," and provides a proof for the following: (1) an equilateral triangle intersects the given circle at the six vertices of a special hexagon, with two non-consecutive sides symmetrically positioned; (2) the construction process forms a regular hexagon inscribed in the given circle.

This paper's solution to the "Triangling A Circle" problem naturally leads to the concept of "regular pentagoning a circle," "regular triangling a circle," "regular hexagoning a circle," and so on.



I. PROOFS OF NEW PROPOSITIONS

I.1 Theorem 1

If there exists an equilateral triangle that has the same area as a given circle (O, r), then 3 sides of the triangle cut the circle at 6 points to form an inscribed hexagon for the circle.

PROOF:

Assume ABC is the equilateral that has the same area as area of a given circle (O, r), then

- the area of ABC is less than the area of the circumscribed equilateral triangle of the circle, obviously (red dashed triangle in Figure 1 below);
- the area of ABC is larger than the area of the inscribed equilateral triangle of the circle, obviously (red dashed smallest triangle in Figure 1 below).

Thus, ABC is located between the 2 red dashed triangles, in Figure 2 below, and three sides of ABC have to cut the circle at 6 points as required. These 6 points are vertices of an inscribed hexagon of the given circle.

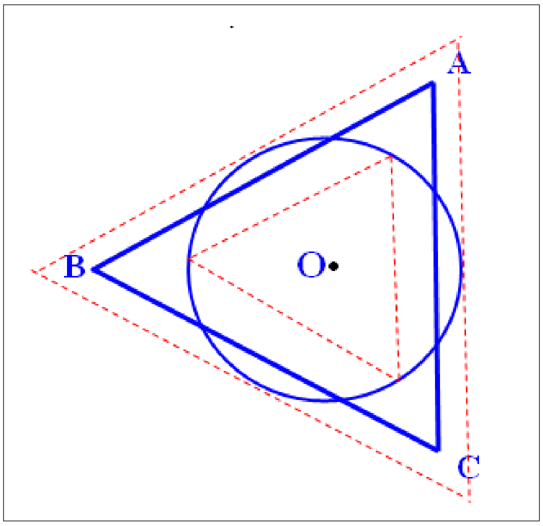


Figure 1: ***The equilateral triangle ABC and the 2 concentric dashed red equilateral triangles of the given circle (O, r).***

I.2 Theorem 2

If there exists an equilateral triangle that has the same area as a given circle (O, r), then the triangle cuts the circumference of the circle (O, r) at 6 points a, b, c, d, e & f, which are 6 vertices of an inscribed regular hexagon of the given circle.

PROOF

Assume there exists an equilateral triangle ABC (blue colour in Figure 2 below) that has the same area πr^2 as the area of a given circle (O, r) and is concentric to the circle.

By Theorem 1 in section I.1 above, this triangle's sides cut the circumference of the circle at 6 points a, b, c, d, e & f. Let a circle (O, R) be the circumscribed circle of the equilateral triangle ABC. Extend chord af of the circle (O, r) to meet the circumference of the circle (O, R) at points E & F, then connect points O & E. From point E, draw a symmetric straight line segment with EF, that cut the circle (O, R) at D. Then, OE is the bisector of angle DEF. Because of the symmetrical property of the lines EF and ED and circle chords af & ed (Figure 2 below), line ED overlaps chord ed in the circle (O, r). Therefore, angle FED is equal to 60° and DEF is also the inscribed equilateral triangle in the circle (O, R). This shows that 2 equilateral triangles inscribed ABC and DEF in the Circle (O, R) are equal, then

Area of triangle ABC = πr^2 = area of triangle DEF (1).

Because (O, r) and (O, R) are the 2 concentric circles, the equilateral triangle Aaf is equal to the equilateral Eef, in other word, 6 equilateral triangles Aaf, Eef, Cde, Ddc, Bbc & Fab are equal. This shows that 6 chords of the circle (O, r) are equal,
ab = bc = cd = de = ef = fa (2).

By (2), the inscribed hexagon abcdef is regular as required.

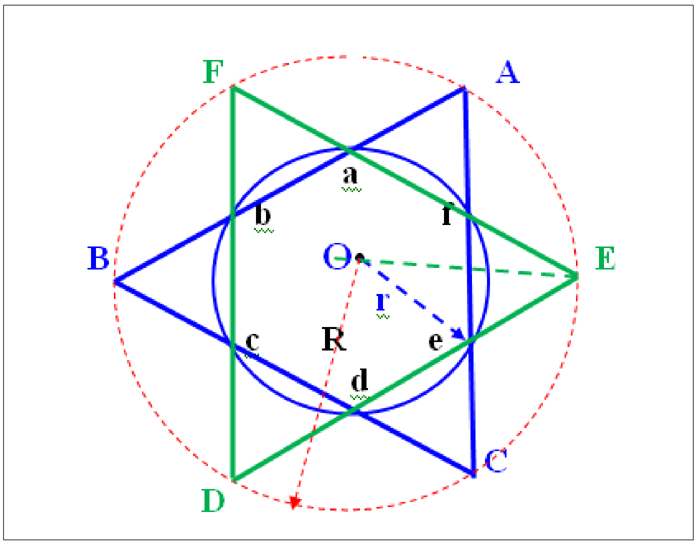
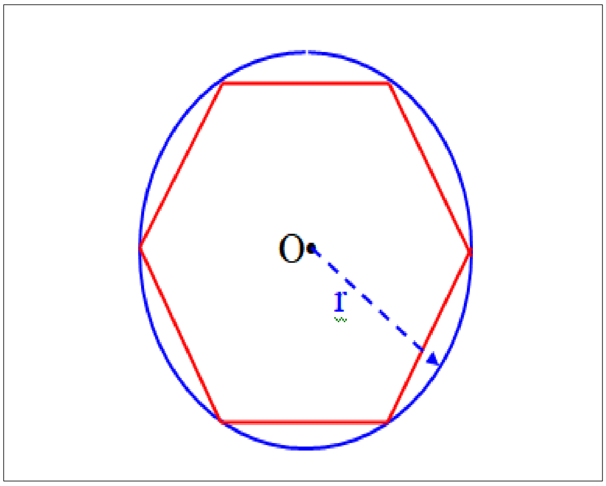


Figure 2: ***abcdefgh (blue and green colours) is the regular Octagon inscribed in Circle (O, r).***

I.3 DEFINITION:

*Given a circle (O, r), area = πr^2, then any regular hexagon inscribed in the circle is defined as a "**TRIANGLING RULER**" of the circle (Figure 3 below).*



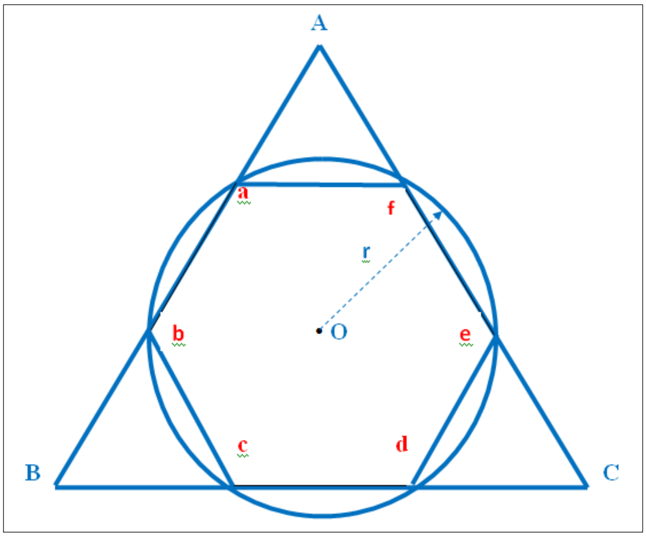


Figure 3: TRIANGLING RULER *abcdef (blue colour) of a given circle (O, r) is the regular hexagon inscribed in the given Circle (O, r).*

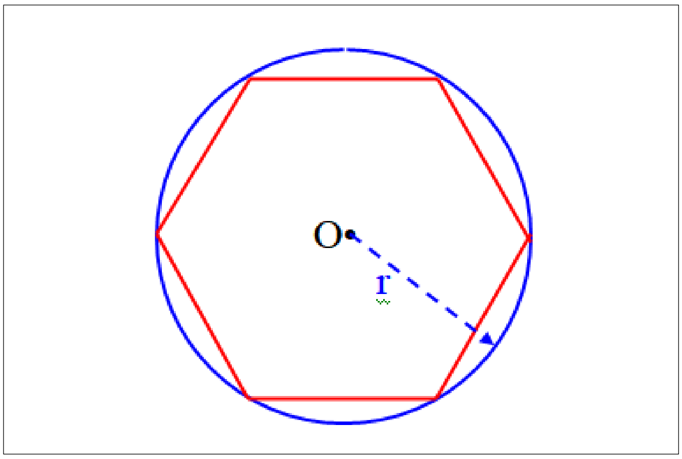
II. REGULAR TRIANGLING A CIRCLE

CORE THEOREM

Given a circle (O, r) then a TRIANGLING RULER of the circle is a mathematics tool to construct an equilateral triangle that has the same area as the area of the given circle, with a straightedge & compass in Euclidean Geometry.

PROOF:

Given a circle (O, r), area πr^2, then by the Definition in section I.3 above, there exists a regular hexagon abcdef, inscribed in the circle, which is the Triangling Ruler, described in the following figure.



Extend 3 sides ab, cd & ef of the hexagon to obtain an equilateral triangle ABC (blue colour in Figure 4 below). By Theorem 2 in Part I above, 3 sides of the equilateral that have the same area πr^2 as the area of the circle, overlap 3 non-consecutive sides of the above hexagon/Triangling Ruler. Therefore, ABC is the solution to the “Regular Triangling the Circle” challenge problem (Figure 4 below).

Note

If one draws the concentric circle (O, R) that is the circumscribed circle of triangle ABC, then the equilateral triangle DEF (red colour in Figure 4 below) is equal to the triangle ABC. Therefore, one Triangling Ruler produces 2 constructable equal solutions to the Regular Triangling the Circle problem with a straightedge & compass in Euclidean Geometry, as required.

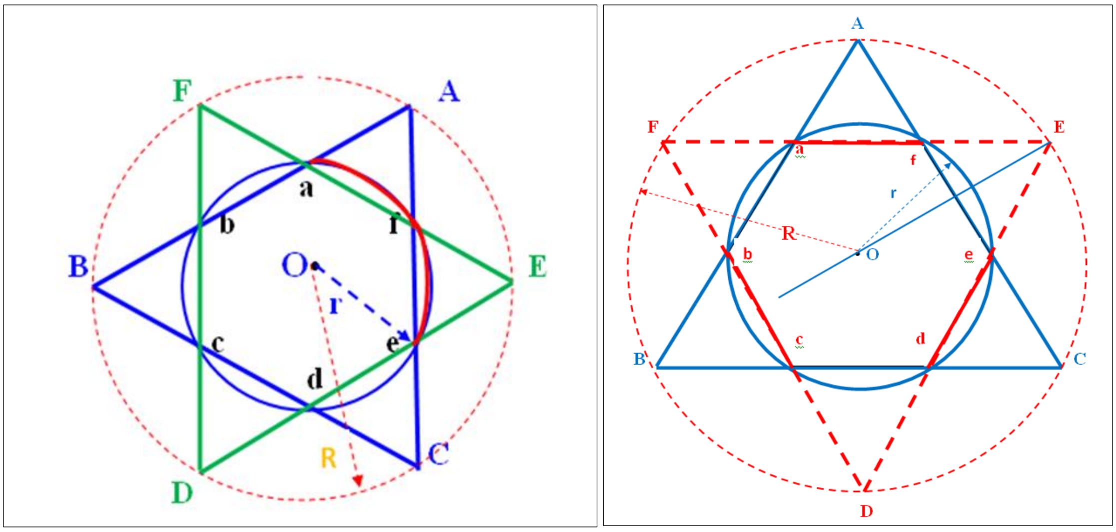


Figure 4: ***abcdef (green & blue / red & bue colours) is the Triangling Ruler or mathematical tool of the given circle (O, r).***

III. DISCUSSION AND CONCLUSION

Can mathematicians use a compass and a straightedge to construct an equilateral triangle of equal area to a given circle, using only a straightedge and compass? The results of my independent research successfully get an exact solution to the ancient Greek “Squaring the Circle” problem challenge showing the idea that “If one can do *“squaring a circle”* then one can do regular *“triangling a circle”* too”. This article paper proves it is certain to construct an equilateral triangle that has the same area pr² as the area of a given circle (O, r), successfully. Also, by [], this equilateral triangle has the same area pr² as the area pr² of a resulting square of my solution to the “Squaring The Circle” published on the SJPMS on (date)….

Our construction method is quite different from approximation and is based on using a straightedge and a compass within Euclidean Geometry.

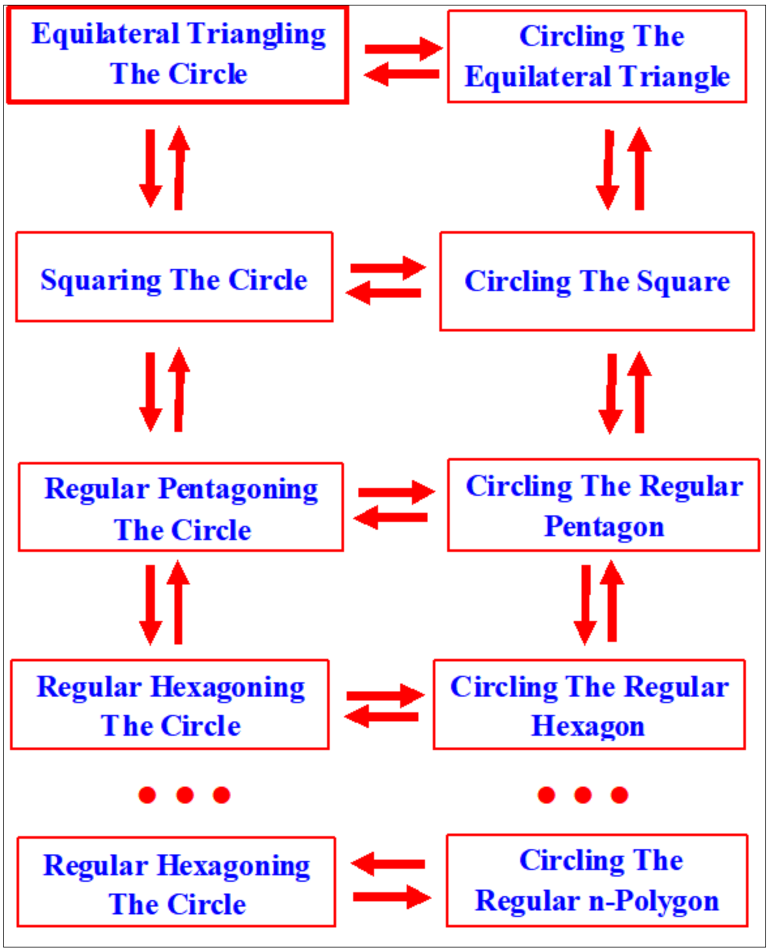
The inspiration for this research arose after the solution to the "Squaring the Circle" problem was published in SJPMS in 2024. If it is possible to square a circle using only a straightedge and compass in Euclidean geometry, the question arises: why is it difficult to triangulate a circle equidistantly, similar to how one squares a circle? In this context, the given circle contains an inscribed hexagon, with three non-consecutive sides extending from it, leading to an equilateral triangle. This equilateral triangle serves as the solution to the new problem of "Regular Triangling of a Circle." The remaining task is to prove that the area of this triangle is exactly equal to the area of the given circle. This paper also introduces a new mathematical tool, called the "Triangling Ruler," and provides a proof for the following statements: (1) A concentric equilateral triangle intersects the given circle at the six vertices of a special hexagon, with two non-consecutive sides symmetrically positioned; (2) This solution construction forms a regular hexagon inscribed in the given circle.

Upstream from this method of exact “Regular Triangling A Circle” above, we can deduce, conversely, to get a new mathematical challenge "Circling an Equilateral Triangle" with a straightedge & a compass only, in Euclidean Geometry. In details, we can describe it as follows:

Given an equilateral triangle with side a & height h, a & h $\subset$R, then use a straightedge & a compass to construct an accurate circle, which has the exact area ½ah. Then, how to solve this new geometry problem is still an open research interest, to get a circle area without using the traditional constant π.

In addition, this research result can be interpreting as further research for a new “REGULAR HEXAGONING A CIRCLE” challenge, with only “a straightedge & compass” in Euclidean Geometry.

Finally derivatives from the Squaring the Circle problem (ancient Greek challenge) for further researches, can be described in the following diagram:



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