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An Exact and Simple Solution to the "Tripling A Cube" Problem Using Straightedge and Compass Only in Euclidean Geometry

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Abstract **Original Research Article**

The problem of tripling a cube consists in constructing, by means of an unmarked straightedge and compass within the framework of Euclidean geometry, a cube whose volume is exactly three times that of a given cube. While the classical Greek problems squaring the circle, trisecting an angle, and doubling the cube have been extensively studied, the analogous problem of tripling a cube has not appeared in the traditional canon [8]. In this paper, we formulate and investigate the problem of tripling a cube for an arbitrary given cube, as a natural extension of prior work on the exact doubling of the cube ("Exact Doubling the Cube with Straightedge and Compass by Euclidean Geometry, by Tran Dinh Son in the Scholars Journal of Physics, Mathematics and Statistics | Volume-12-Issue-02, in 2025, Sch J Phys Math Stat | 24-32, DOI : https://doi.org/10.36347/sjpms.2025.v12i02.001".). We present a construction-based approach that yields an exact solution using only classical Euclidean tools, without recourse to transcendental quantities such as π. The method is grounded in elementary geometric principles, though its full development requires a nontrivial sequence of constructions. The results establish a novel straightedge-and-compass construction for achieving a cube of triple volume and introduce a systematic framework referred to as the Tripling a Cube method that enables precise and reproducible solutions. This work contributes a new perspective to classical geometric construction problems and extends the scope of constructability within Euclidean geometry.

Keywords: tripling a cube, three times a cube, tripling cube, multiple cubes, make a cube larger 3 times, cube multiplication into a larger cube.

I. INTRODUCTION

For over three millennia, three classical problems from ancient Greek geometry - "Squaring The Circle", "Doubling The Cube", and "Trisecting an Angle" - have tested the ingenuity of mathematicians. Formulated under the strict constraint of using only an unmarked straightedge and compass, these problems were ultimately declared unsolvable within the standard framework of Euclidean construction. In the nineteenth century, foundational impossibility results by Pierre Wantzel and Ferdinand Von Lindemann, employing algebraic field theory and transcendental number theory, appeared to settle the matter: Wantzel's 1837 theorem ruled out arbitrary angle trisection and cube duplication by straightedge-and-compass methods, and Lindemann's proof of the transcendence of π (1882) made classical squaring of the circle impossible. [4,5,2].

Although these algebraic impossibility theorems are rigorous within their respective algebraic and arithmetic frameworks, they depart from the purely constructive spirit of the ancient geometric problems. Classical Greek geometry is fundamentally geometric and constructive in nature - eschewing algebraic or transcendental reasoning - and this difference in perspective motivates a re-evaluation of the ancient challenges strictly from within Euclidean geometric methods. [2,3,18,19]. Related publications include the paper "Exact Solution to the Squaring the Circle Problem" (SJPMS, 2024), which presents a construction claimed to produce a square whose area equals that of a given circle solely by Euclidean means and challenges the conventional interpretation of Lindemann's impossibility result. The inverse problem, "Circling the Square," was also published in 2024, and these methods were extended to a new problem, "Circling the Regular Hexagon." The latter problem apparently not previously treated in the literature considers the construction of a circle concentric with a given regular hexagon that has the same area. The proposed method identifies a regular dodecagon whose twelve vertices lie on the same circle and on the extended sides of the hexagon; inscribing this dodecagon by straightedge and compass yields a circle whose area equals that of the hexagon. To facilitate this approach a specialized instrument, the Regular Dodecagon Ruler, is introduced. [6,7,16,17].

Starting from accepted premises, without proof, one uses deductive reasoning to arrive at theorems and corollaries. With different premises, we develop different mathematical systems. For example, the premise "from a point outside a line, only one parallel line can be drawn to the given line" leads to Euclidean geometry. If we assume that no parallel lines can be drawn from that point, we enter the realm of Riemannian geometry. Alternatively, Lobachevskian geometry assumes that an infinite number of parallel lines can be drawn through that point [30,31]. No scientific theory lasts forever, but specific research and discoveries continuously build upon each other. The three classic ancient Greek mathematical challenges likely referring to are “Doubling the Circle”, “Trisecting an Angle” and “Squaring the Circle”, all famously proven Impossible under strict compass-and-straightedge constraints, by Pierre Wantzel (1837) using field theory and algebraic methods, then also by Ferdinand von Lindemann (1882) after proving π is transcendental [2-5]. These original Greek challenges remain impossible under classical rules since their proofs rely on deep algebraic/transcendental properties settled in the 19th century. Recent claims may involve reinterpretations or unrelated advances but do overturn the impossible conclusions above [6-22]. Among these, the "Squaring A Circle" problem and related problems involving π have captivated both professional and amateur mathematicians for millennia.

Beyond technical constructions, this research contributes to the philosophical discussion about mathematical truth and the role of framework in determining what is “possible” or “impossible.” While impossibility theorems are decisive within the logical systems and assumptions that underpin them, the results presented here suggest that alternative constructive approaches - consistent with the original spirit of Euclidean geometry - may produce exact solutions previously regarded as unattainable. This perspective aligns with Karl Popper’s philosophy of science, in which knowledge is provisional and open to falsification by new evidence. [1].

In addition, the investigations were guided by an aesthetic and philosophical principle inspired by Lao Tzu’s aphorism from the Tao Te Ching: ***“The Great Tao is simple, very simple”*** (大道至简) [32]. Emphasizing simplicity and concentric structure, the author develops exact geometric constructions not only for angle trisection but also for the classical problems of squaring the circle and doubling the cube, relying exclusively on geometric reasoning. [6-10].

This article asserts that the problem of constructing a cube with triple the volume of a given cube can be resolved within Euclidean geometry using only a straightedge and compass. The construction presented here follows the classical methods of analysis and synthesis: in the analytic phase the construction is assumed and worked backward to reduce the problem to known solvable configurations, and in the synthetic phase the solution is reconstructed step by step from the initial data [33].

The construction derived from analysis and synthesis yields a complete and rigorous solution to the problem of tripling the cube; although intricate, the method demonstrates that exact tripling is achievable by classical means.

The previous research further leverages this foundation to produce a substantially simplified, exact construction for tripling a cube - again, strictly within the compass-and-straightedge paradigm. During this work a novel geometric implement - the Head-cut Pyramid - was devised. This instrument enables rapid and precise tripling of any given cube and represents a practical advancement in constructive geometry.

The remainder of this paper presents the analytic reductions, synthetic constructions, proofs of correctness, and descriptions of the “tripling a cube” solution & the Head-cut Pyramid. Detailed diagrams and stepwise compass-and-straightedge procedures are provided to demonstrate the exactness and reproducibility of the constructions.

II. PROPOSITION

Let C_1 be a cube of side length $a > 0$, and let C_2 be a cube of side length $a\sqrt[3]{3}$, constructed so that C_1 and C_2 are concentric (C_2 = $3C_1$).
Denote by $R = \frac{C_2}{C_1}$ the region contained in C_2but exterior to C_1 (see **Figure 1**).

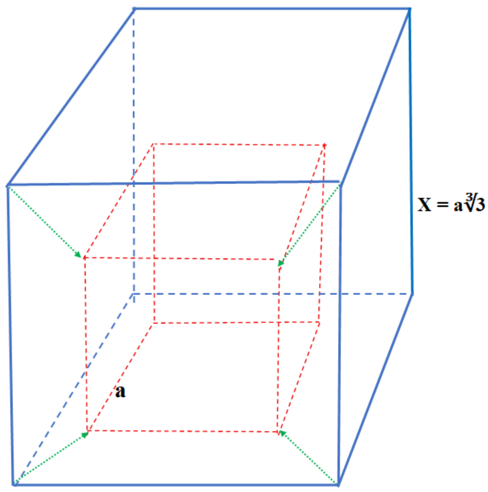


Figure 1: Two concentric cubes: the inner cube C_1 of side length a, and the outer cube C_2 of side length $a\sqrt[3]{3}$.

The region R admits a natural decomposition into finitely many congruent solids, each of which can be described in terms of truncated pyramidal geometry. This observation motivates the following definition.

II.1 **Head-cut Pyramid Definition**:

A *head-cut pyramid* is defined as the frustum of a regular square pyramid; that is, a solid obtained by intersecting a regular square pyramid with a plane parallel to its base and removing the portion containing the apex (see Figure 2).

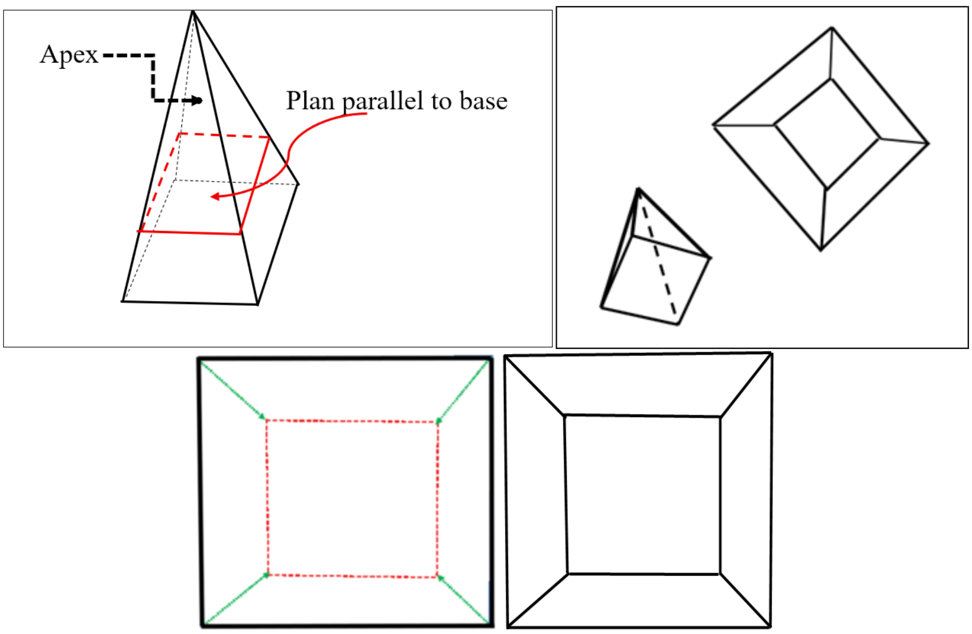


Figure 2: Head-cut Pyramid shape

II.2 **Head-cut Pyramid Theorem**:

Let two cubes be concentric in $\mathbb{R}^3$, with the smaller cube of volume a^3 strictly contained within the larger. Then the six head-cut pyramids (truncated pyramids), formed between corresponding faces of the two cubes, are congruent and equal.

Proof:

Let C_{out} and C_{in} denote the outer and inner cubes, respectively, sharing a common centre. Denote by F_i^{out} and $F_i^{\text{in}}(i = 1, \dots, 6)$ the corresponding parallel faces of C_{out} and C_{in}.

Join the vertices of C_{out} to the corresponding vertices of C_{in}. This partitions the region between the two cubes into six head-cut pyramids (truncated pyramids), each bounded by a pair of parallel square faces F_i^{out} and F_i^{in}, together with four lateral faces.

Since the cubes are concentric, the perpendicular distance between each pair F_i^{out} and F_i^{in} is constant for all i. Hence, all six head-cut pyramids (truncated pyramids) have equal height.

Moreover, all faces of a cube are congruent squares and equal correspondingly. Therefore, for each i, the base F_i^{out} has the same area for all pyramids, and similarly the top face F_i^{in} has the same area for all pyramids.

Thus, each head-cut pyramid (truncated pyramid) has identical height and congruent parallel faces. It follows that the six head-cut pyramids (truncated pyramids) are congruent and equal.

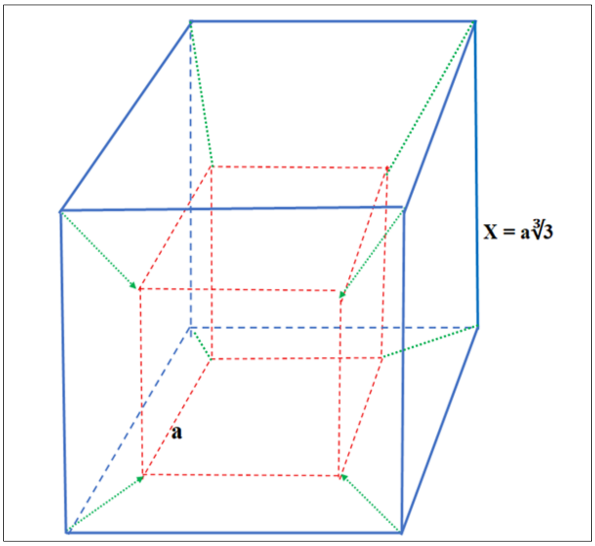


Figure 3: Six congruent and equal head-cut pyramids (truncated pyramids, see the green arrows) surrounding the cube of volume a^3

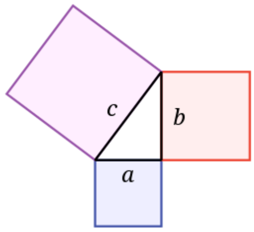
II.3 Theorem 01:

Given a UNIT LENGTH u, then

a. The exact lengths of √2, √3 and √4 are constructive in algebraic geometry with a compass and a straightedge. and,

b. The exact length Y = $\sqrt[3]{3} = \frac{3+\sqrt{105}}{6}$ is also constructive in algebraic geometry with a compass and a straightedge.

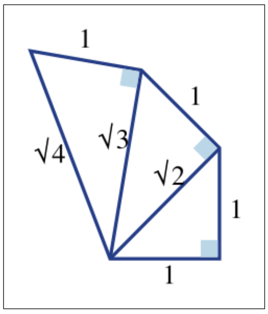
Note and reminding on the Pythagorean theorem

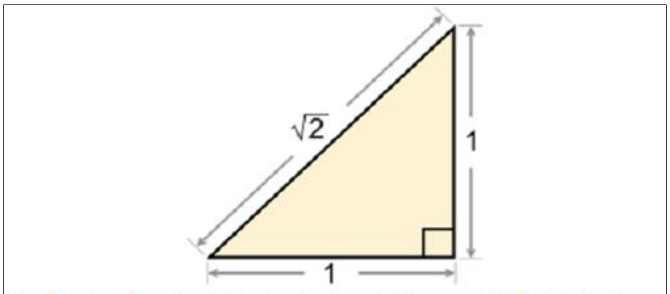


In mathematics, the Pythagorean theorem or Pythagoras's theorem is a fundamental relation in Euclidean geometry between the three sides of a right triangle. It states that the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares on the other two sides. The theorem can be written as an equation relating the lengths of the sides a, b and the hypotenuse c, sometimes called the Pythagorean equation:

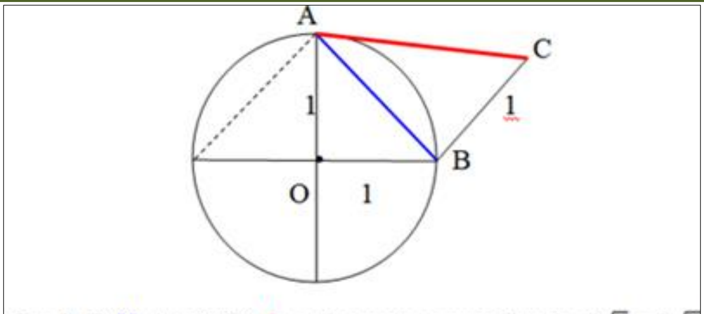
$a^2 + b^2 = c^2$.

Greek philosopher Pythagoras, born around 570 BC. The theorem has been proved numerous times by many different methods – possibly the most for any mathematical theorem. The proofs are diverse, including both geometric proofs and algebraic proofs, with some dating back thousands of years (see Images below).

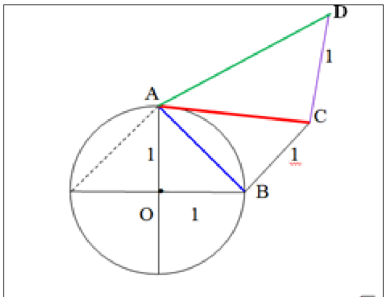




Note: ***The above figure is a right-angle triangle marked 'square root of 2' at the hypotenuse.***



Length AB (blue) and AC (red) are the exact geometrical lengths of $\sqrt{2}$ and $\sqrt{3}$



Length AD (green) is the exact length of $\sqrt{4}$

End of Note.

PROOF:

Geometric Construction of Successive Square Roots and a Derived Segment

a.- Let a unit length $u = 1$ be given. All constructions, in this article paper, are performed using only a straightedge and compass.

Construction of $\sqrt{2}$ and $\sqrt{3}$

- Construct a circle with centre O and radius 1.
- Let A and B be end points of two perpendicular diameters of the circle.
- Then triangle AOB is right-angled at O, with:

$\mathrm{OA} = \mathrm{OB} = 1.$

By the Pythagorean theorem:

$\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2 = 1^2 + 1^2$

Hence: $\mathrm{AB} = \sqrt{2}.$

➢Extend the construction by forming a right triangle ABC such that:

$\mathrm{BC} = 1$ and $\mathrm{AB} = \sqrt{2}.$

Applying the Pythagorean theorem again:

$\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2 = (\sqrt{2})^2 + 1^2$

Therefore:

$\mathrm{AC} = \sqrt{3}.$

Iterative Construction up to $\sqrt{105}$

➢At point C, construct a line perpendicular to AC. On this perpendicular, mark a point D such that (see illustrated images below):

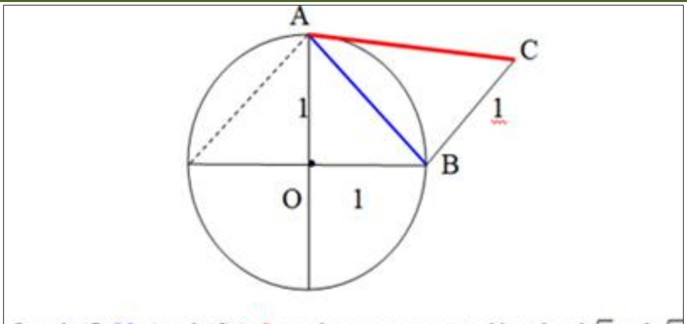
CD = 1

Then triangle ACD is right-angled at C, and:

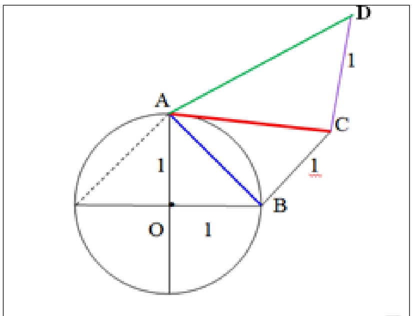
$\mathrm{AD}^2 = \mathrm{AC}^2 + \mathrm{CD}^2 = (\sqrt{3})^2 + 1^2$

Hence:

$\mathrm{AD} = \sqrt{4}.$



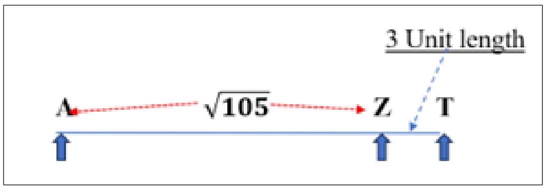
: Length AB (blue) and AC (red) are the exact geometrical lengths of $\sqrt{2}$ and $\sqrt{3}$



: Length AD (green) is the exact length of $\sqrt{4}$.

➢ Repeating this procedure inductively each time erecting a perpendicular of unit length at the latest point produces a sequence of segments:
$\sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{105}.$

b.- After 104 such steps, one obtains a segment:
$AZ = \sqrt{105}.$



Construction of the line Segment AT $= \sqrt{105} + 3$
➢Extend segment AZ by adding 3 unit lengths to obtain (image above):
$AT = \sqrt{105} + 3.$

Construction of the Segment Y $= \sqrt[3]{3} \;= \frac{3+\sqrt{105}}{6}$
➢Construct a line segment AU such that (see Figure 4 below):
$AU = 6.$

➢Join U to T.
➢Divide AU into six equal unit segments.
➢Through each division point, construct lines parallel to UT.
➢Let S be the intersection point corresponding to one unit from A.

Then, by similarity of triangles (intercept theorem), the line segment AS satisfies:
$$AS = \frac{AT}{6} = \frac{3 + \sqrt{105}}{6}.$$

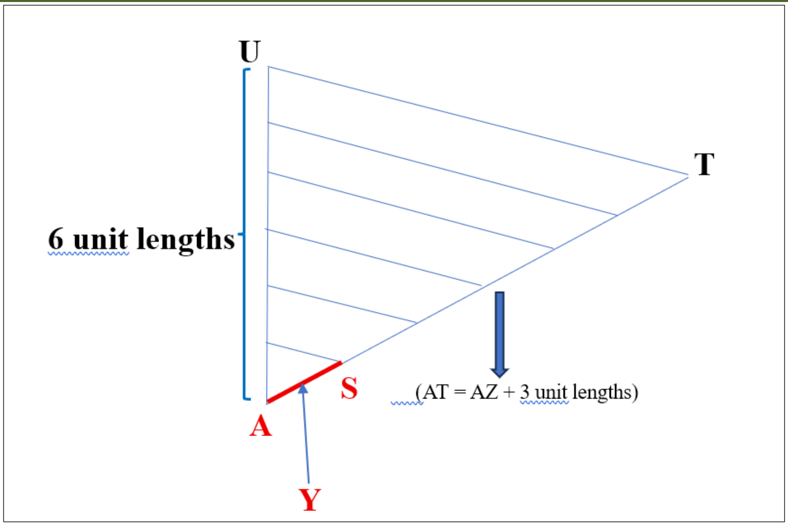


Figure 4: Length Y = AS = $\sqrt[3]{3} = \frac{3+\sqrt{105}}{6} = \sqrt[3]{3}$

Thus, this construction provides:

- A classical straightedge-and-compass method to generate successive square roots,
- A systematic extension up to $\sqrt{105}$,
- And a proportional construction yielding:

$$Y = \sqrt[3]{3} = \frac{3+\sqrt{105}}{6}.$$

II.4 Core Theorem (Tripling Cube Theorem):

The tripling cube volume $3a^3$, from a given cube volume a^3, is constructive exactly and simply with straightedge and compass in Euclidean Geometry.

Proof:

Remind that "Tripling A Cube" problem is also the same as "multiple a cube by 3".

Let a cube of volume $3a^3$ and side X be a tripled cube from a given cube of volume a^3 and side a. Then place the given cube concentric inside the tripled cube. By Head-cut Pyramid Theorem (section II.2 above), the height H and the volume V of any head-cut pyramid is calculated as follows:

$$H = \frac{X-a}{2} = \frac{a\sqrt[3]{3}-a}{2} = \frac{a(\sqrt[3]{3}-1)}{2}$$

$$V = \frac{a(\sqrt[3]{3}-1)}{2} x \frac{X^2+a^2}{2} = \frac{X^2a\sqrt[3]{3}-aX^2+\sqrt[3]{3}a^3-a^2}{4} \qquad (1)$$

as average area of two bases of a head-cut pyramid is $\frac{X^2+a^2}{2}$.

There are six head-cut pyramids, surrounding the given cube, occupying a volume of $(3a^3-a^3) = 2a^3$. Then, volume V of one head-cut pyramid is

$$\frac{2a^3}{6} = \frac{a^3}{3} \qquad (2)$$

By (1) & (2), one get

$$\frac{(X-a)(X^2+a^2)}{4} = \frac{a^3}{3}$$

$$\frac{X^3+a^2X-aX^2+a^3}{4} = \frac{a^3}{3}$$

Replace X by $a\sqrt[3]{3}$ to get

$$\frac{3a^3+a^2a\sqrt[3]{3}-a(a\sqrt[3]{3})^2+a^3}{4} = \frac{a^3}{3}$$

Let denote $\sqrt[3]{3}$ = Z, then $\frac{3a^3+a^2aZ-a(a^2Z^2)+a^3}{4} = \frac{a^3}{3}$

becomes

$\frac{3a^3+a^3Z-a^3Z^2+a^3}{4} = \frac{a^3}{3}$

$\frac{-a^3Z^2+Z+4a^3}{4} = \frac{a^3}{3}$

$-3a^3Z^2 + 3a^3Z + 12a^3 = 4a^3$

$-3a^3Z^2 + 3a^3Z + 8a^3 = 0 \qquad (3)$

Then (3) is a quadratic equation in term of variable $Z = \sqrt[3]{3}$.
In solving this equation for Z, one gets:

$\Delta = 9a^6 - 4(-3a^3)(8a^3) = 9a^6 + 96a^6 = 105a^6$

$\sqrt{\Delta} = a^3\sqrt{105}$

$Z = \frac{-b \pm \sqrt{\Delta}}{2a}$

$Z = \frac{3+\sqrt{105}}{6} > 0$ **or** $\frac{3-\sqrt{105}}{6} < 0$, if $a \neq 0$

$Z = \frac{-3+\sqrt{105}}{-6} < 0$ **or** $Z = \frac{-3-\sqrt{105}}{-6} > 0$

Therefore, the answer for this solution is

$Z = \sqrt[3]{3} = \frac{3+\sqrt{105}}{6}$

By Theorem 1 above (section II.3), the result $Z = \sqrt[3]{3} = \frac{3+\sqrt{105}}{6}$ is constructive geometrically with straightedge & compass, into length Z (Figure 5 below). Because a is a given real number, of which one can be used to multiply length Z into the length $aZ = a\sqrt[3]{3} = a\frac{3+\sqrt{105}}{6}$, such that the resulting cube with volume $3a^3$ can be constructive exactly and simply (Figure 5 below).

Thus, the "tripling a cube" problem or "multiplying a cube by 3" problem is solve exactly and simply using only straightedge & compass in Euclidean Geometry as required.

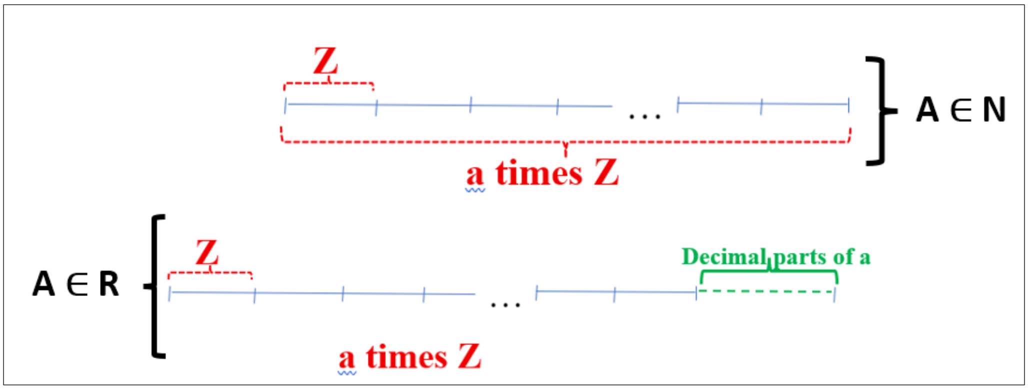


Figure 5: Length a times Z for the construction of the result cube with volume 3a³

III. MATERIALS & METHODS

The materials and methods include straightedge, compass, ANALYSIS Method and SYNTHESIS Method, within the scope of Algebraic Geometry and Pure Geometry.

IV. DISCUSSION & CONCLUSION

IV.1 Discussion

Innovation Cannot Thrive When Constrained by Traditional Theories

Generating new mathematics arises from the creative synthesis of existing theories, rigorous formalism, and innovative physical or computational insights. One can extend known frameworks to produce novel theorems, apply advanced models such as four-dimensional topology or B-matrix statistics to reveal previously hidden relationships, or reinterpret classical equations like Schrödinger's or relativistic formulations to uncover new structures and distributions. At its core, new mathematics emerges when deep theoretical insight meets cross-disciplinary vision, producing results that are both logically consistent and generative of further inquiry. This approach demonstrates that mathematics is far from closed; it continues to expand wherever curiosity, abstraction, and experimentation intersect.

In the past, those who entered the construction industry typically gained some understanding of its history. One notable figure is Joseph Monier (1823–1906), the inventor of reinforced concrete. He first presented his invention at the Paris Exhibition in 1867 and was granted the world's first patent for reinforced concrete. Subsequently, he received additional patents for reinforced concrete pipes, tanks, beams, and other applications. The first reinforced concrete bridge was also built according to his design. However, Monier's invention was not initially recognized by construction engineers and leading experts in France and around the world. Bound by conventional theories that treated steel and concrete as separate materials, and lacking knowledge of their combined potential, they doubted the durability of the composite material. Furthermore, due to Monier's status as a common individual rather than an academic or professional insider, his work was largely disregarded. As a result, meaningful application of his invention was delayed until the late 19th and early 20th centuries *(nearly 30 years delay!).* Nevertheless, reinforced concrete eventually became one of the greatest innovations in human history, revolutionizing the construction industry. This breakthrough is attributed to Monier, a self-taught inventor whose ideas ultimately prevailed despite initial resistance. His success was made possible by a few individuals in the engineering community who recognized the invention's value and either acquired the rights or continued to develop it. Today, it is widely acknowledged that without reinforced concrete, it would be impossible to construct skyscrapers, strong bridges, modern highways with overpasses and underpasses, and the vast infrastructure required for large contemporary cities. A similar example can be seen in the invention of Blockchain technology by Satoshi Nakamoto an individual whose identity remains unknown due to a deliberate choice to remain anonymous. Like Monier, Nakamoto's work has had a profound impact on the world, despite coming from outside traditional academic or institutional frameworks. These examples demonstrate that strict adherence to established theories can hinder creativity and innovation. True progress often originates from those willing to think beyond conventional boundaries and from those who dare to explore uncharted territory. Another example of great invention is the Blockchain technology of Satoshi Nakamoto, a person who has a name but no one knows who he is.

In the past, knowledge was often considered scientific if it could be confirmed through specific evidence or experiments. However, Karl Popper, in his book Logik der Forschung (The Logic of Scientific Discovery), published in 1934, demonstrated that a fundamental characteristic of scientific hypotheses is their ability to be proven wrong (falsifiability). Anything that cannot be refuted by evidence is temporarily regarded as true until new evidence emerges. For instance, in astronomy, the Big Bang theory is widely accepted, but in the future, anyone who discovers a flaw in this theory will be acknowledged by the entire physics community. Furthermore, no scientific theory lasts forever; rather, it is specific research and discoveries that continually build upon one another [1].

Moreover, PEOPLE MAY STAND STILL – BUT THE EARTH DOES NOT: In 1851, in the nave of the Panthéon, Léon Foucault conducted a quiet yet revolutionary experiment. He suspended from the dome a pendulum 67 meters in length. As it oscillated, observers noticed that the plane of its swing gradually rotated over time. The pendulum itself did not change direction; rather, the Earth was rotating beneath it.

With this simple but profound demonstration, Foucault provided direct, visible evidence that the Earth spins on its axis. Prior to this experiment, the heliocentric model proposed by Nicolaus Copernicus and later supported by Galileo Galilei had already established the theory of Earth's rotation. However, these conclusions rested primarily on mathematical reasoning and astronomical observation. Foucault sought something more immediate: empirical proof accessible to all.

No complex equations were required. No debate was necessary. One needed only to stand and observe. For the first time in history, people could witness with their own eyes that they inhabited a moving planet.

Foucault did more than demonstrate a physical phenomenon; he transformed humanity's perception of its place in the universe. He made the invisible visible. In an age of intellectual contention, he chose demonstration over argument, evidence over rhetoric.

This principle extends beyond physics. In contemporary society, opinions are abundant, and debates are constant. Many claim to possess ideas, ambitions, and potential. Yet ideas alone do not alter reality. The world does not revolve around assertions; it advances through action.

Foucault's pendulum did not persuade because he spoke about it it persuaded because it moved.

If one believes oneself capable of meaningful achievement, the path forward is not endless discussion but deliberate creation. As the ancient philosopher Laozi (Lao Tzu) expressed, "The Great Tao is simple." Simplicity, however, does not imply passivity. It calls for clarity of purpose and decisive effort: build something tangible, write with substance, develop expertise, initiate a project and substantiate claims with results [32].

When work is visible and measurable, validation becomes unnecessary. People will not ask whether you are capable; they will observe what you have created and draw their own conclusions.

We live on a planet in constant motion. The Earth continues to rotate, indifferent to hesitation. So too do opportunities evolve and pass. The essential question, then, is not whether the world moves but whether we move with it.

The lesson drawn from great figures is not merely admiration of their achievements. It is the recognition that decisive action transforms theory into experience, and potential into reality. It looks like the Newton's apple showed people the visible gravity force. The opportunity to begin remains always today, as this article transformed an invisible "tripled cube" from a given cube, into a constructive "tripled cube" visibly.

The "Doubling the Cube" problem refers to the ancient Greek problem of constructing a cube with double the volume of a given cube, using only a straightedge and compass. The problem dates back to at least the 5th century BC and was one of the three famous unsolved problems of ancient Greek mathematics, alongside the "Trisecting an Angle" and the "Squaring the Circle" problems [6,9]. Solution to the "Doubling the Cube" problem was published [10]. Nowaday, this "Tripling A Cube" problem also refers to the ancient Greek problem of constructing a cube with double the volume of a given cube, using only a straightedge and compass.

My past research result objectively presents a provable construction of generating a length of magnitude; as the geometrical solution for the ancient classical problem of doubling the volume of a given cube, and published [10]. The "Doubling the Cube" problem, which has challenged mathematicians since the time of the ancient Greeks, is precisely solved by the ANALYTICS method to concentrically locate a given cube of volume a^3 in its double cube with volume $2a^3$, side $a\sqrt[3]{2}$. In other words, I did succeed the concentric location for the given cube a^3 inside the goal cube $2a^3$ to solve exactly the problem with a straightedge and a compass; then this solution was published in [10].

The impossibility proof of doubling a cube, published by mathematician Wantzel, was based on three-dimensional cubic extensions in abstract algebra, an approach that entirely shifted the problem to solid geometry from its Greek‟s definition in plane geometry, and therefore the algebraic statement of impossibility has no geometrical validity [4] & [5].

I don't know how many creative ideas are accidentally stifled like the so-called IMPOSSIBILY above. Many people cannot overcome this "Thinking Trap", not only binding themselves but also hindering others. "Thinking Trap" is the word used to refer to the stuckness of people who have the habit of constantly establishing a state of impossibility for their will. They always react to strange, new, unusual things; to things that they are not confident in or feel threatened by saying no to them. The human brain operates to provide information and arguments that its owner desires. Therefore, when it is determined to be impossible, one's thoughts and actions will be in the direction of trying to prove oneself right - that is, making oneself and related people see that it is impossible. In extreme cases, people even try to sabotage the work of those who are proving it is possible. How disastrous is such an extreme case that falls on people in power? But these are common cases in society.

People with low self-esteem often fall into the trap of thinking. They not only think they can't do it, but also doubt the ability of others. If they are powerful people, they will give themselves the right to ensure safety. As a result, they will stifle creativity. Only people with confidence and a scientific attitude can overcome the trap of thinking. If we see difficulties, we should point them out and analyse them scientifically, not instill or impose low self-esteem. Doing so will cause a decline in will, and never dare to overcome things that are bigger than ourselves.

This is evident from the fact that no two facets of a cube can share all four vertices from two different planes. However, according to this study result, the impossible imprecise classification should not be extended to geometry so that the irrationality definition was stated as "algebraic irrationality is not a constructible number of the geometry". The possibility to solve geometrically the coefficient constant $\sqrt[3]{2}$ to an exact precision is proved. This study paper also presents a geometrically certain method under the set restrictions of Euclidean geometry (in the sense that, all presented constructions have been reduced to the Euclidean postulates of practical geometry), by the construction of the relation as depicted in the justification section in Sch J Phys Math Stat | 24-32, *SJPMS 13/02/2025,* https://doi.org/10.36347/sjpms.2025.v12i02.001, file:///C:/Users/buung_000/Documents/Downloads/SJPMS_122_24-32%20(3).pdf [10].

The above solution, using only a straightedge and compass, referred to as the "Tripling A Cube", did not previously arise in classical geometric construction. It emerged only after I resolved the long-standing challenge of "Doubling The Cubbe" problem and published the solution in Sch J Phys Math Stat | 24-32, *SJPMS* *13/02/2025,* https://doi.org/10.36347/sjpms.2025.v12i02.001, file:///C:/Users/buung_000/Documents/Downloads/SJPMS_122_24-32%20(3).pdf , [10].

IV.2 CONCLUSION

Most mathematicians and mathematics enthusiasts accept that the three classical Greek problems Squaring the Circle**,** Doubling the Cube**, and** Trisecting an Angle - are impossible to solve using only a straightedge and compass. This consensus largely stems from the work of Pierre Wantzel (1837), who employed field theory and algebraic methods to prove the impossibility of certain geometric constructions [4] & [5]. However, it is important to recognize that "Squaring the Circle" is fundamentally a geometric construction problem, and the algebraic approach may not fully address the nuances of Euclidean geometry. Further support for the impossibility of squaring the circle came from Ferdinand von Lindemann's proof in 1882 that π is transcendental. From this, it is commonly concluded that since π cannot be constructed using a finite sequence of straightedge and compass steps, it is impossible to square the circle in the classical sense [2] & [3].

However, it is worth considering a different perspective. This article can construct a tripled cube from an arbitrary cube by placing the given cube inside and concentric with the tripled cube whose edge is unknown. To do so, the idea of shapes surrounding the given cube and adjacent to the 6 square faces of the tripled cube must be considered and verified and defined: The Head-cut Pyramid. I calculate the Volume of a Head-cut Pyramid to get a cubic equation, of which when substituting the volume of 1/6 of the space bounded by the 12 faces of the 2 cubes, we obtain a quadratic equation with unknown being $\sqrt[3]{3}$. I solve this quadratic equation to have $\sqrt[3]{3} = \frac{3+\sqrt{105}}{6}$, which is number can be express into a length using straightedge & compass. Therefore the side of the resulting cube (the tripled cube) is identified as $a\sqrt[3]{3} = a\frac{3+\sqrt{105}}{6}$, exactly in term of length by means of straightedge & compass.

I hold a different point: *I believe I have constructed a valid solution to the "Tripling A Cube" problem using only a straightedge and compass, in accordance with the classical constraints.* This belief strengthens my resolve as I pursue a solution to the "Multiply a cube by 4 times", "Multiply a cube by 5 times" problems, and so on … The techniques of geometrical analysis and synthesis are instrumental in this effort.

Moreover, further investigation solutions can be researched for the "multiply a cube by n times, n ∈ Z = {1, 2, 3, …}", in the near future.

It is difficult to realise why the "Triple A Cube" challenge above has not existed before this published article, meanwhile its solution within the classic Euclidean Geometry is proved simply as above.

V. AN OPEN AREA FOR RESEARCH

The Tripling Cube Theorem (in section II.3 above), applied for tripling the cube of volume a^3 into the cube with volume $3a^3$, certainly converted from "its cubic equation to its quadratic equation successfully", in order to have a precise geometrical length constructed by straightedge & compass. Therefore, the new problem of *"converting a cubic equation to a quadratic equation, equivalently"* is a possible open research area in algebraic geometry.

Conflicts of Interest
The author declares that there is no conflict of interest regarding the publication of this paper.

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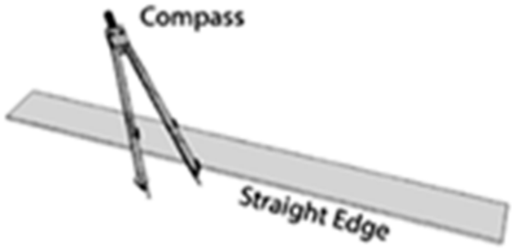
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