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Research Article

Integer Points on the Homogeneous Cone $7x^2$-$2y^2$=$10z^2$

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Abstract: The ternary quadratic $7x^2 - 2y^2 = 10z^2$ representing a homogeneous cone is analysed for its non-zero distinct integral points. A few interesting properties among the solutions and polygonal numbers are presented. Given an integrer solution, six different triples of integers generating infinitely many integer solutions are exhibited.
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INTRODUCTION

The ternary homogeneous quadratic diophantine equation offers an unlimited field for research because of their variety[1-2]. For an extensive review of various problems one may refer [3-11]. In this context, one may also see [12-16] for integer points satisfying special three dimensional graphical representations. This communication concerns with yet another interesting ternary quadratic equation $7x^2 - 2y^2 = 10z^2$ representing homogeneous cone for determining its infinitely many non-zero integer solutions. A few interesting properties among the solutions and special numbers are presented. Also given an integer solution, six different triples of integers generating infinitely many integer solutions are exhibited

NOTATION USED

. $t_{m,n}$ - polygonal number of rank n with size m

METHOD OF ANALYSIS

The ternary quadratic eqution studied for its non distinct integer solutions is given by

$$7x^2 - 2y^2 = 10z^2 \quad (1)$$

We illustrate below the different patterns of integer solutions to (1)

Pattern I

Introduction of the linear transformations

$$x = X + 2T \ , \ y = X + 7T \quad (2)$$

in (1) leads to

$$X^2 - 14T^2 = 2Z^2 \quad (3)$$

which can be written as

$$X^2 - 16T^2 = 2(Z^2 - T^2) \quad (4)$$

Rewrite (4) in the form of ratio as

$$\frac{X + 4T}{Z + T} = \frac{2(Z - T)}{X - 4T} = \frac{A}{B} \quad \text{where } B \neq 0 \quad (5)$$

(5) is equivalent to the following two equations

$$BX + T(4B - A) - AZ = 0, \ - AX + T(4A - 2B) + 2BZ = 0 \quad (6)$$

By the method of cross multiplication, we get

$$X = 12A^2 - 4AB\,, T = A^2 - 2B^2 \quad (7)$$

$$Z = -A^2 - 2B^2 + 8AB \quad (8)$$

From (2) and (7), the corresponding non-zero integer values of x and y are given by

$$x(A,B) = 14A^2 - 4B^2 - 4AB\,, y(A,B) = 19A^2 - 14B^2 - 4AB \quad (9)$$

Thus (8) and (9) represent the distinct integer point on the cone (1).

PROPERTIES

1. $x(A,1) + z(A,1) - t_{28,A} \equiv 0 \pmod{16}$
2. $6\left[x(1,4-6n-n^{2)} + y(1,4-6n-n^2) + t_{14,4-6n-n^2}\right]$ is a nasty number 3.

$y(1,4n-3) + z(1,4n-3) + t_{547,n} - 5 \equiv 0 \pmod{145}$

4. $y(1,4n-3) + z(1,4n-3) + 544t_{3,n-1} \equiv -5 \pmod{144}$

NOTE

Equation (3) can be also be written in the following ways

1. $\frac{X+4Z}{2(T+Z)} = \frac{7(T-Z)}{X-4Z} = \frac{A}{B}$ 2. $\frac{X-4Z}{2(T+Z)} = \frac{7(T-Z)}{X+4Z} = \frac{A}{B}$

3. $\frac{X+4Z}{7(T+Z)} = \frac{2(T-Z)}{X-4Z} = \frac{A}{B}$ 4. $\frac{X-4Z}{7(T+Z)} = \frac{2(T-Z)}{X+4Z} = \frac{A}{B}$

5. $\frac{X+4Z}{T+Z} = \frac{14(T-Z)}{X-4Z} = \frac{A}{B}$ 6. $\frac{X-4Z}{T+Z} = \frac{14(T-Z)}{X+4Z} = \frac{A}{B}$

7. $\frac{X+4Z}{14(T+Z)} = \frac{T-Z}{X-4Z} = \frac{A}{B}$ 8. $\frac{X-4Z}{14(T+Z)} = \frac{T-Z}{X+4Z} = \frac{A}{B}$ where $B \neq 0$

Proceeding as above , different choices of integer solutions to (1) are obtained

PATTERN:2

Introducing the linear transformations

$$x = 4X + 2\alpha \pm 4\beta\,, y = 4X + 7\alpha \pm 14\beta \,,\quad Z = \alpha \pm 14\beta$$

in (1),and performing a few algebra ,the corresponding two sets (I,II) of non- zero distinct integer solutions to (1) are represented as follows.

SET:I

$$x(p,q) = 168p^2 + 2q^2 + 8pq$$
$$y(p,q) = 308p^2 - 3q^2 + 28pq$$
$$z(p,q) = 28p^2 - q^2 - 28pq \quad (10)$$

PROPERTIES:

1. $x(1,q) - y(1,q) - 8t_{3,p} \equiv -12 \pmod{16}$
2. $x(1,n) + y(1,n) + 4t_{3,n} \equiv 14 \pmod{34}$
3. $x(p,1) + y(p,1) + z(p,1) + t_{226,p} \equiv -4 \pmod{159}$

SET:II

$x(p,q)=168p^2+2q^2-8pq$

$y(p,q)=308p^2-3q^2-28pq$

$z(p,q)=28p^2-q^2+28pq$

PROPERTIES:

1. $y(p,1)-x(p,1)-5t_{58,p}\equiv 2(\bmod 7)$
2. $x(1,z)-z(1,z)-t_{8,z}\equiv 4(\bmod 34)$
3. $x(1,n)+z(1,n)-2t_{3,n}\equiv 7(\bmod 21)$

REMARKS:

Instead of (2), one may also consider the linear transformations to be

$$x=X-2T\ ,\ y=X-7T$$

Following the procedures presented in patterns I&II, the corresponding 3 sets (III,IV,V) of integer solutions to (1)are as follows .

SET:III

$x(A,B)=10A^2+4B^2-4AB$

$y(A,B)=5A^2-4AB+14B^2$

$z(A,B)=-A^2-2B^2+8AB$

SET:IV

$x(p,q)=56p^2+6q^2-8pq$

$y(p,q)=-84p^2+11q^2-28pq$

$z(p,q)=28p^2-q^2-28pq$

SET:V

$x(p,q)=56p^2+6q^2+8pq$

$y(p,q)=-84p^2+11q^2+28pq$

$z(p,q)=28p^2-q^2+28pq$

GENERATION OF SOLUTIONS

Let (x_0,y_0,z_0) be the initial solution of (1).Then each of the following triples of non –zero distinct integers based on (x_0,y_0,z_0) also satisfies (1).

TRIPLE:I $(3^n x_0, y_n, z_n)$

here $$y_n=\frac{3^{n-1}}{2}\left[(5+(-1)^n)y_0+((-1)^n 5-5)z_0\right]$$

$$z_n=\frac{3^{n-1}}{2}\left[((-1)^n-1)y_o+(1+(-1)^n 5)z_0\right]$$

TRIPLE:II ($x_n, 3^n y_0, z_n$ **)**

here $$x_n=3^{n-1}\left[(10-7(-1)^n)x_0+((-1)^n 10-10)z_0\right]$$

$$z_n = 3^{n-1}\left[(7-7(-1)^n)x_0 + ((-1)^n 10 - 7)z_0\right]$$

TRIPLE:III $(x_n, y_n, 3^n z_0)$

where

$$x_n = 3^{n-1}\left[(24+(-1)^n(-21))x_0 + ((-1)^n 12 - 12)y_0\right]$$
$$y_n = 3^{n-1}\left[(42+(-1)^n(-42))x_0 + ((-1)^n 24 - 21)y_0\right]$$

TRIPLE:IV $(3^n x_0, y_n, z_n)$

where

$$y_n = \frac{1}{2\sqrt{5}}\left[\sqrt{5}((2+i\sqrt{5})^n + (2-i\sqrt{5})^n)y_0 + i5((2+i\sqrt{5})^n - (2-i\sqrt{5})^n)z_0\right]$$
$$z_n = \frac{1}{2\sqrt{5}}\left[-i((2+i\sqrt{5})^n - (2-i\sqrt{5})^n)y_0 + \sqrt{5}((2+i\sqrt{5})^n + (2-i\sqrt{5})^n)z_0\right]$$

TRIPLE:V $(x_n, 3^n y_0, z_n)$

in which

$$x_n = \frac{1}{2\sqrt{70}}\left[\sqrt{70}\left((17+2\sqrt{70})^n + (17-2\sqrt{70})^n\right)x_0 + 10\left((17+2\sqrt{70})^n - (17-2\sqrt{70})^n\right)z_0\right]$$
$$z_n = \frac{1}{2\sqrt{70}}\left[7\left((17+2\sqrt{70})^n + (17-2\sqrt{70})^n\right)x_0 + \sqrt{70}\left((17+2\sqrt{70})^n - (17-2\sqrt{70})^n\right)z_0\right]$$

TRIPLE:VI (x_n, y_n, z_0)

in which

$$x_n = \frac{1}{2}\left[(15+4\sqrt{14})^n + (15-4\sqrt{14})^n\right]x_0 + \frac{1}{\sqrt{14}}\left[(15+4\sqrt{14})^n - (15-4\sqrt{14})^n\right]y_0$$
$$y_n = \frac{\sqrt{14}}{2}\left[(15+4\sqrt{14})^n - (15-4\sqrt{14})^n\right]x_0 + \frac{1}{2}\left[(15+4\sqrt{14})^n + (15-4\sqrt{14})^n\right]y_0$$

OBSERVATION

Let x, y represent non- zero distinct positive integer solutions to (1).Denote $x+y$ by P and y by q. Treat p,q as the generators of the pythagorean triangle $s(\alpha,\beta,\gamma)$ where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2$. It is seen that $s(\alpha,\beta,\gamma)$ is such that

1) $6\gamma + \beta - 7\alpha \equiv 0(\text{mod}10)$
2) $7\gamma - 8\alpha + \frac{4A}{p} \equiv 0(\text{mod}10)$

CONCLUSION

In this paper ,we have obtained inifinitely many integer points satisfying the cone $7x^2 - 2y^2 = 10z^2$.As ternary quadratic diophantine equations are rich in variety, one may consider other choices of ternary quadratic diodhantine equations and search for their integer solutions.

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