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Research Article

Heine Theorem and Its Application

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Abstract: In the first part of this paper, it firstly takes $x \to x_0$ and $x \to \infty$ as an example. It presents that the unary function has Heine theorem of the normal limit existence and promotion form, and also Heine theorem of the abnormal limit existence. Secondly, it presents that Heine theorem of normal unilateral limit existence has conditions with stronger forms. Finally, it shows that the multivariate function has Heine theorem of normal and abnormal limit existence. In the second part, it lists some application of Heine theorem.
Keywords: Heine theorem, functional limit, sequence limit

I. Heine theorem

Theorem 1[1] Set f to have definition within $U^o(x_0)$. $\lim_{x \to x_0} f(x) = A \Leftrightarrow \forall \{x_n\} \subset U^o(x_0)$, and $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n) = A$ is valid.

Similarly, there is Heine theorem of functional limit existence when $x \to x_0^-, x \to x_0^+$, $x \to \infty$, $x \to -\infty$ and $x \to +\infty$. For instance:

Theorem 2 Set f to have definition within $\{x : |x| > a\}$. $\lim_{x\to\infty} f(x) = A \Leftrightarrow \forall \{x_n\} \subset \{x : |x| > a\}$, and $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} f(x_n) = A$ is valid.

Theorem1 and theorem2 can be generalized as:

Theorem $1'$ Set f to have definition within $U^o(x_0)$. $\lim_{x\to x_0} f(x)$ exists $\Leftrightarrow \forall \{x_n\} \subset U^o(x_0)$, and $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n)$ exists.

Theorem $2'$ Set f to have definition within $\{x : |x| > a\}$, $\lim_{x\to\infty} f(x)$ exists $\Leftrightarrow \forall \{x_n\} \subset \{x : |x| > a\}$, and $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} f(x_n)$ exists.

When $x \to x_0^-, x \to x_0^+, x \to -\infty$ 及 $x \to +\infty$, Heine theorem of the functional limit existence has stronger form of the condition. For instance:

Theorem3[2]-[4] Set f to have definition within $U_+^o(x_0)$. $\lim_{x\to x_0^+} f(x)$ exists $\Leftrightarrow$ As for any decreasing sequence $\{x_n\} \subset U_+^o(x_0)$, and $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n)$ exists.

Theorem4[2]-[4] Set f to have definition within $(a,+\infty)$. $\lim_{x\to+\infty} f(x)$ exists $\Leftrightarrow$ As for any increasing sequence $\{x_n\}\subset(a,+\infty)$, and $\lim_{n\to\infty} x_n=+\infty$, $\lim_{n\to\infty} f(x_n)$ exists.

With regard to the unary function, there are 18 kinds of Heine Theorem of abnormal limit existence. For instance:

Theorem 5 Set f to have definition within $U^o(x_0)$. $\lim_{x\to x_0} f(x)=\infty \Leftrightarrow \forall\{x_n\}\subset U^o(x_0)$, and $\lim_{n\to\infty} x_n=x_0$, $\lim_{n\to\infty} f(x_n)=\infty$ is valid.

Theorem6 Set f to have definition within $(a,+\infty)$. $\lim_{x\to+\infty} f(x)=-\infty \Leftrightarrow \forall\{x_n\}\subset(a,+\infty)$, and $\lim_{n\to\infty} x_n=+\infty$, $\lim_{n\to\infty} f(x_n)=-\infty$ is valid.

Theorem7 [5]Set the definition domain of the multivariate function $z=f(P)$ to be D, P_0 is the accumulation point of D. $\lim_{P\to P_0} f(P)=A \Leftrightarrow \Leftrightarrow \forall\{P_n\}\subset U^o(P_0)\text{I } D$, and $\lim_{n\to\infty} P_n=P_0$, $\lim_{x\to x_0} f(x)=A$ is valid.

Theorem8 Set the definition domain of the multivariate function $z=f(P)$ to be D, P_0 is the accumulation point of D. $\lim_{P\to P_0} f(P)=\infty \Leftrightarrow \Leftrightarrow \forall\{P_n\}\subset U^o(P_0)\text{I } D$, and $\lim_{n\to\infty} P_n=P_0$,
$\lim_{n\to\infty} f(P_n)=\infty$ is valid.

Similarly, with regard to the multivariate function, there is Heine Theorem of $\lim_{P\to P_0} f(P)=-\infty$ and $\lim_{P\to P_0} f(P)=+\infty$.

Corollary 1 Set f to have definition within $U(x_0)$. f is continuous at $x_0 \Leftrightarrow \forall\{x_n\}\subset U(x_0)$, and $\lim_{n\to\infty} x_n=x_0$, $\lim_{n\to\infty} f(x_n)=f(x_0)$ is valid.

Corollary 2 Set the definition domain of the multivariate function $z=f(P)$ is D, $P_0\in D$, and P_0 is the accumulation point of D. f is continuous at $P_0 \Leftrightarrow \forall\{P_n\}\subset U(P_0)\text{I } D$, and $\lim_{n\to\infty} P_n=P_0$, $\lim_{n\to\infty} f(P_n)=f(P_0)$ is valid.

APPLICATION OF HEINE THEOREM

Example 1 Set f to be a monotonous and bounded function within $U_+^0(x_0)$, show that $\lim_{x\to x_0^+} f(x)$ exists.

Prove Set f to be the monotone increasing function within $U_+^0(x_0)$.

Arbitrarily select a decreasing sequence $\{x_n\}\subset U_+^0(x_0)$, and $\lim_{n\to\infty} x_n=x_0$, then, $\{f(x_n)\}$ is bounded decreasing sequence. From the monotonous and bounded theorem of sequence , it turns out that $\lim_{n\to\infty} f(x_n)$ exists. From Theorem 3, it turns out that $\lim_{x\to x_0^+} f(x)$ exists.

Example 2 Set f to have definition within $(a,+\infty)$, show that $\lim_{x\to+\infty} f(x)$ exists $\Leftrightarrow$
$\forall\varepsilon>0,\exists X>a,\ \forall\ x_1,x_2\in(X,+\infty),\ |f(x_1)-f(x_2)|<\varepsilon.$

Prove only prove the sufficiency. Arbitrarily select a sequence $\{x_n\} \subset (a, +\infty)$, and $\lim_{n\to\infty} x_n = +\infty.$

Apply the condition: $\forall \varepsilon > 0, \exists X > a, \ \forall \ x_1, x_2 \in (X, +\infty), \ |f(x_1) - f(x_2)| < \varepsilon.$

From the above X, $\exists N \in N^+$, when $n > N$, $x_n \in (X, +\infty)$. Therefore, when $n, m > N$, $x_n, x_m \in (X, +\infty)$, $|f(x_n) - f(x_m)| < \varepsilon.$

Namely, $\forall \varepsilon > 0, \ \exists N \in N^+$, When $n, m > N$, $|f(x_n) - f(x_m)| < \varepsilon.$

From the Cauchy criterion with the sequence limit existence, it turns out that $\lim_{n\to\infty} f(x_n)$ exists. From theorem 4, it turns out that $\lim_{x\to+\infty} f(x)$ exists.

Example 3 Set function f, g and h has definition when $|x| > a (a \geq 0)$, and meet:

(1) When $|x| > a$, $g(x) \leq f(x) \leq h(x)$;

(2) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = A,$

show that $\lim_{x\to\infty} f(x) = A.$

Prove Arbitrarily select a sequence $\{x_n\}$, meet $|x_n| > a \ (n = 1, \ 2, \Lambda)$, and $\lim_{n\to\infty} x_n = \infty.$

As $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = A,$ according to theorem 2, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = A$ is valid.

Besides, due to $g(x_n) \leq f(x_n) \leq h(x_n), n = 1,2,\Lambda$, according to the squeeze rule of the sequence limit, it turns out that $\lim_{n\to\infty} f(x_n) = A.$

From theorem 2, $\lim_{x\to\infty} f(x) = A.$

Example 4 Let $f(x) = \frac{1}{x}\sin\frac{1}{x}$, show that when $x \to 0$, f does not have the normal limit and the abnormal limit.

Prove record $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}, x_n' = \frac{1}{2n\pi}, n = 1,2,\Lambda$, and then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n' = 0.$

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(2n\pi + \frac{\pi}{2}) = +\infty,$ From theorem 1, when $x \to 0$, f does not have the normal limit. .

$\lim_{n\to\infty} f(x_n') = \lim_{n\to\infty} 0 = 0,$ From theorem5, when $x \to 0$, f does not have the abnormal limit.

Example 5 Obtain the limit $\lim_{n\to\infty} \frac{\ln\sin(\arctan n)}{(\pi - 2\arctan n)^2}$.

Solve Set $x_n = \arctan n, n = 1,2,\Lambda$. So, $\lim_{n\to\infty} x_n = \frac{\pi}{2}.$

$$\lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2} = -\frac{1}{4}\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin x(\pi - 2x)} = -\frac{1}{4}\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{(\pi - 2x)} = -\frac{1}{8}\lim_{x\to\frac{\pi}{2}} \sin x = -\frac{1}{8}.$$

From theorem1, $\lim_{n\to\infty} \frac{\ln\sin(\arctan n)}{(\pi - 2\arctan n)^2} = \lim_{n\to\infty} \frac{\ln\sin x_n}{(\pi - 2x_n)^2} = \lim_{x\to\frac{\pi}{2}} \frac{\ln\sin x}{(\pi - 2x)^2} = -\frac{1}{8}.$

Example 6 Let $f(P) = f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Prove Select $P_n(\frac{1}{n},0), P_n^{/}(0,\frac{1}{n}), n = 1,2,\Lambda$. So, $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} P_n^{/} = O.$

$$\lim_{n\to\infty} f(P_n) = \lim_{n\to\infty} f(\frac{1}{n},0) = 1;$$

$$\lim_{n\to\infty} f(P_n^{/}) = \lim_{n\to\infty} f(0,\frac{1}{n}) = -1.$$

From theorem 7, it turns out that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Example 7 Set function $z = f(P)$ to be continuous in the region D. The point set E is the dense point set within D . If $f(P) \equiv A$ on E , show that $\forall P \in D$, $f(P) \equiv A.$

Prove Arbitrarily select $P_0 \in D,$ as f is continuous at P_0, so $\lim\limits_{P\to P_0} f(P) = f(P_0).$ As E is the dense point set in D , it is feasible to select the point sequence $\{P_n\} \subset E,$ and $\lim\limits_{n\to\infty} P_n = P_0.$ From Corollary 2, $\lim\limits_{n\to\infty} f(P_n) = f(P_0).$

Besides, as $f(P_n) \equiv A,$ $\lim\limits_{n\to\infty} f(P_n) = A.$ According to the uniqueness of the sequence limit, it turns out that $f(P_0) = A.$

From the arbitrariness of the point $P_0 : \forall P \in D$, $f(P) \equiv A.$

CONCLUSION

Heine theorem reveals the internal relationship between the disperse change and continuous change of the variables, and it is the bridge for communicating the functional limit and the sequence limit. Besides, it plays a vital role in the limit theory. To apply Heine theorem can prove the nature theorem and existence theorem of the functional limit. Besides, it is feasible to get the sequence limit through the functional limit, and deny the functional limit existence.

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