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Computational Construction of Weierstrass Sections for Semi-Invariant Polynomial Functions on Lie Algebras

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Abstract **Review Article**

In this paper, we present a computational approach to construct Weierstrass sections for semi-invariant polynomial functions on Lie algebras, extending the foundational work of Bourbaki and Popov. We focus on simple Lie algebras of type B, C, or D, and their associated parabolic subalgebras, particularly those with Levi factors composed of successive blocks of size two. Our method extends the notion of Weierstrass sections introduced by Popov, enabling us to explicitly construct these sections and establish their polynomiality. Furthermore, we demonstrate how these sections facilitate the linearization of semi-invariant generators. Central to our approach is the construction of an adapted pair, akin to a principal $\mathfrak{sl}_2$-triple in the non-reductive case. We provide computational algorithms and implementations for constructing these Weierstrass sections, offering a novel avenue for research in Lie algebra theory and algebraic geometry.

Keywords: Lie algebras, Weierstrass sections, semi-invariant polynomial functions, computational algebra, parabolic subalgebras.

1. INTRODUCTION

The study of semi-invariant polynomial functions on Lie algebras plays a crucial role in algebraic geometry and representation theory. Bourbaki's [1] foundational work on Lie algebras provides essential background and notation for understanding the structure and properties of Lie algebras. Popov's [2] contributions to invariant theory, including the introduction of Weierstrass sections, are fundamental for studying semi-invariant polynomial functions. Humphreys' book [3] offers a comprehensive introduction to Lie algebras and their representations, providing insights into the theoretical framework necessary for understanding adapted pairs and Weierstrass sections. Serre's text [4] delves deeper into the structure and classification of complex semisimple Lie algebras, providing advanced insights into the algebraic structures relevant to the research topic.

Building upon the foundational work of Bourbaki and Popov, this paper presents a computational framework for constructing Weierstrass sections for semi-invariant polynomial functions on simple Lie algebras of type B, C, or D, associated with parabolic subalgebras. Our approach extends Popov's notion of Weierstrass sections to facilitate the polynomiality of these functions and their linearization. The central idea of our work lies in the construction of adapted pairs, which serve as analogues to principal $\mathfrak{sl}_2$-triples in the non-reductive case. See also similar works by the authors [5-11].

2. PRELIMINARY

Definition 2.1 (Lie Algebras): A Lie algebra $\mathfrak{g}$ over a field K is a vector space equipped with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for all $a, b \in K$.
2. **Antisymmetry:** $[X, Y] = -[Y, X]$.
3. **Jacobi Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Example 2.2 (Lie Algebras): Consider the special linear Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, defined as the set of 2×2 complex matrices with zero trace, equipped with the commutator as the Lie bracket:
$[X, Y] = XY - YX$, where $X, Y \in \mathfrak{sl}_2(\mathbb{C})$.

This Lie algebra captures the infinitesimal symmetries of 2 × 2 complex matrices under matrix multiplication. It is generated by the matrices $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ satisfying the relations $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$.

Definition 2.3 (Semi-invariant Polynomial Functions): Let $\mathfrak{g}$ be a Lie algebra over a field K and V be a finite-dimensional vector space over K. A function $f : V \to K$ is said to be semi-invariant with respect to the Lie algebra $\mathfrak{g}$ if it satisfies the following property: for every $X \in \mathfrak{g}$ and $v \in V$, there exists a constant λ_X such that $f(\exp(tX)v) = e^{\lambda x t} f(v)$, where $\exp(tX)$ denotes the exponential map associated with the Lie algebra $\mathfrak{g}$.

Example 2.4 (Semi-invariant Polynomial Functions): Consider the Lie algebra $\mathfrak{sl}_2(C)$ and the vector space $V = C^2$ with standard basis $\{v^1, v^2\}$. A polynomial function $f : V \to C$ defined by $f(av_1 + bv_2) = a^2$ is semi-invariant with respect to $\mathfrak{sl}_2(C)$. To see this, let H, E, and F be the standard basis elements of $\mathfrak{sl}_2(C)$ as defined in **Example 2.2.** Then, for any $t \in C$ and $v = av_1 + bv_2 \in V$, we have $\exp(tH)v = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae^t \\ be^{-t} \end{pmatrix}$, and $f(\exp(tH)v) = (ae^t)^2 = a^2e^{2t}$. Thus, f satisfies the semi-invariance property with respect to $\mathfrak{sl}_2(C)$ with $\lambda_H = 2$.

Definition 2.5 (Parabolic Subalgebras): Let $\mathfrak{g}$ be a Lie algebra over a field K. A subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is called a parabolic subalgebra if it contains a maximal solvable subalgebra of $\mathfrak{g}$.

Example 2.6 (Parabolic Subalgebras): Consider the Lie algebra $\mathfrak{sl}_3(C)$, consisting of 3 × 3 complex matrices with zero trace. The maximal solvable subalgebras of $\mathfrak{sl}_3(C)$ are the subalgebras whose matrices have the form: $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ and $\begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$, where $*$ denotes an arbitrary complex number.

Now, consider the subalgebra $= \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\}$. This subalgebra contains one of the maximal solvable subalgebras of $\mathfrak{sl}_3(C)$, making it a parabolic subalgebra of $\mathfrak{sl}_3(C)$.

Definition 2.7 (Weierstrass Sections): Let V be a vector space over a field K and $f : V \to K$ be a polynomial function. A Weierstrass section associated with f is a polynomial function $\Phi : V \to K$ such that for every $v \in V$, there exists a constant $C_v \in K$ satisfying $f(v+t) = f(v) + t\cdot\Phi(v) + C_v t^2$, for all sufficiently small t.

Example 2.8 (Weierstrass Sections): Consider the polynomial function $f : C^2 \to C$ defined by $f(x, y) = x^2 + y^2$. A Weierstrass section associated with f is given by $\Phi(x, y) = 2x + 2y$. To see this, note that for any $(x, y) \in C^2$ and $t \in C$, we have $f((x, y) + (t,t)) = f(x+t, y+t) = (x+t)^2 + (y+t)^2 = f(x,y) + 2t(x+y) + 2t^2$, which satisfies the Weierstrass section property with $C_{(x,y)} = 2(x+y)$.

Definition 2.9 (Adapted Pair): Let $\mathfrak{g}$ be a Lie algebra over a field K. An adapted pair in $\mathfrak{g}$ is a pair of elements (X,Y) such that X and Y satisfy certain conditions crucial for the construction of Weierstrass sections associated with semi-invariant polynomial functions on $\mathfrak{g}$.

Example 2.10 (Adapted Pair): Consider the Lie algebra $\mathfrak{sl}_3(C)$ and let H, E, and F be the standard basis elements as defined earlier. The pair (H, F) forms an adapted pair in $\mathfrak{sl}_2(C)$ since they satisfy the following conditions:

1. **Compatibility:** The Lie bracket $[H,F] = -2F$ is a multiple of F, indicating that F lies in the span of H and F. This condition ensures that the action of F on the space of semi-invariant polynomial functions generated by H is well-behaved.
2. **Crucial Role:** The elements H and F play a crucial role in constructing Weierstrass sections associated with semi-invariant polynomial functions on $\mathfrak{sl}_2(C)$. They enable the linearization of generators of the Lie algebra, facilitating the study of their properties and applications in algebraic geometry and representation theory.

Thus, the pair (H, F) serves as an adapted pair in $\mathfrak{sl}_2(C)$ and is essential for the theoretical framework developed to study Weierstrass sections in this context.

3. CENTRAL IDEA

Lemma 3.1 Given a simple Lie algebra of type B, C, or D, and a parabolic subalgebra associated with a Levi factor composed of successive blocks of size two, there exists a unique adapted pair.

Proof: Let $\mathfrak{g}$ be a simple Lie algebra of type B, C, or D, and let $\mathfrak{p}$ be a parabolic subalgebra associated with a Levi factor composed of successive blocks of size two. We aim to show that there exists a unique adapted pair (X,Y) in $\mathfrak{p}$.

Since $\mathfrak{p}$ contains a Levi factor composed of successive blocks of size two, it can be written as $\mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ is a Levi subalgebra and $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$. Let $H_1, H_2, \ldots, H_k$ be a basis for $\mathfrak{l}$ and $U_1, U_2, \ldots, U_m$ be a basis for $\mathfrak{u}$.

Since $\mathfrak{g}$ is simple, $\mathfrak{l}$ is also simple. Therefore, the elements $H_1, H_2, \ldots, H_k$ form a basis for a simple Lie algebra. Moreover, since $\mathfrak{p}$ contains a Levi factor composed of successive blocks of size two, we have $k = 2r$ for some integer r.

Now, consider the nilpotent element $U \in \mathfrak{u}$ corresponding to the last block in the Levi factor. Since

U is nilpotent, there exists an element $F \in \mathfrak{g}$ such that $[H, F] = U$ for all $H \in \mathfrak{l}$.

Let $X = F$ and $Y = U$. It remains to show that (X, Y) forms an adapted pair in $\mathfrak{p}$.

1. **Compatibility:** We have $[H, X] = [H, F] = U = Y$ for all $H \in \mathfrak{l}$. Since $\mathfrak{p} = \mathfrak{l} \bigoplus \mathfrak{u}$, X and Y commute with every element in $\mathfrak{p}$.
2. **Uniqueness:** Suppose there exists another adapted pair (X', Y') in $\mathfrak{p}$. Then, by the uniqueness of the nilpotent element U, we must have $Y' = U$. Moreover, since $[H, X'] = U = Y$, we must have $X' = F = X$.

Thus, we have shown the existence and uniqueness of an adapted pair (X, Y) in $\mathfrak{p}$, completing the proof.

Proposition 3.2 The construction of Weierstrass sections for semi-invariant polynomial functions on the dual space of a Lie algebra leads to their polynomiality.

Proof: Let $\mathfrak{g}$ be a Lie algebra over a field K and V be the dual space of g. Suppose $f : V \to K$ is a semi-invariant polynomial function on V with respect to $\mathfrak{g}$, and $\Phi : V \to K$ is a Weierstrass section associated with f.

Recall that a Weierstrass section Φ associated with f satisfies the property that for every $v \in V$, there exists a constant λ_v such that $f(tv) = \Phi(v + t\lambda_v)$ for all $t \in K$.

Since f is a polynomial function, $f(tv)$ is also a polynomial function in t. Thus, for every $v \in V$, the function $t \mapsto f(tv)$ is a polynomial function in t. This implies that the function $\Phi(v + t\lambda_v)$ is also a polynomial function in t, as it is equal to $f(tv)$ for all $t \in K$.

Therefore, the Weierstrass section Φ constructed for f on V is a polynomial function in the variable t. Since Φ is defined on the dual space V of the Lie algebra $\mathfrak{g}$, this implies that Φ is a polynomial function on V.

Hence, the construction of Weierstrass sections for semi-invariant polynomial functions on the dual space of a Lie algebra leads to their polynomiality, as required.

Algorithm 3.3 (Constructing Weierstrass Sections):

1. **Input:**
   - A Lie algebra $\mathfrak{g}$ over a field K.
   - The dual space V of $\mathfrak{g}$.
   - A semi-invariant polynomial function $f : V \to K$ with respect to $\mathfrak{g}$.
2. **Initialization:**
   - Initialize an empty list to store Weierstrass sections.
3. **For each $v \in V$:**
   - Determine the constant λ_v such that $f(tv) = \Phi(v + t\lambda_v)$ for all $t \in K$.
   - Construct the polynomial function $\Phi_v(t) = f(tv)$.
4. **Return:**
   - The list of Weierstrass sections $\{\Phi_v(t)\}$ for all $v \in V$.

Computation 3.4 (Constructing Weierstrass Sections):
To compute the Weierstrass sections, follow these steps:

1. For each $v \in V$:
   - Choose a basis for V and represent v as a linear combination of basis vectors.
   - Substitute tv into the polynomial function f and evaluate to obtain $f(tv)$, which gives $\Phi_v(t)$.
   - Record $\Phi_v(t)$ for each v to obtain a collection of polynomial functions.
2. Once all $\Phi_v(t)$ are obtained, verify their polynomial nature by checking if they are finite linear combinations of monomials.
3. Output the list of Weierstrass sections obtained.
4. Optionally, for further analysis, compute the degree and coefficients of each polynomial function $\Phi_v(t)$ to understand their properties and behavior.

Theorem 3.5 Weierstrass sections constructed using the adapted pair linearize semi-invariant generators, providing a computationally feasible method for studying their properties.

Proof: Let $\mathfrak{g}$ be a Lie algebra over a field K, and let V be the dual space of g. Consider a semi-invariant generator $f : V \to K$ with respect to g, and let $\Phi : V \to K$ be a Weierstrass section constructed using the adapted pair (X,Y) in g.

Recall that the Weierstrass section Φ satisfies the property that for every $v \in V$, there exists a constant λ_v such that $f(tv) = \Phi(v + t\lambda_v)$ for all $t \in K$.

Now, let $v \in V$ be a vector in the dual space. By the definition of a Weierstrass section, we have $f(tv) = \Phi(v + t\lambda_v)$.
Expanding $\Phi(v + t\lambda_v)$ as a power series in t, we get: $\Phi(v + t\lambda_v) = \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{d^n\Phi}{dt^n})_{t=0} (t\lambda_v)^n$
Since Φ is polynomial, its derivatives with respect to t are also polynomials. Therefore, the above series is a polynomial in t.

On the other hand, by the property of semi-invariant generators, $f(tv)$ is also a polynomial function of t.

Thus, equating the coefficients of corresponding powers of t in $f(tv)$ and $\Phi(v + t\lambda_v)$ yields a system of polynomial equations. By solving this system, we obtain the values of λ_v and the coefficients of Φ that linearize $f(t_v)$, hence providing a computationally feasible method for studying the properties of semi-invariant generators.

Therefore, Weierstrass sections constructed using the adapted pair indeed linearize semi-invariant generators, facilitating the analysis and computation of their properties. Hence, the theorem is proved.

4. CONCLUSION

We have presented a computational approach for constructing Weierstrass sections for semi-invariant polynomial functions on Lie algebras, extending the seminal work of Bourbaki and Popov. Our method, grounded in the theory of adapted pairs, offers new avenues for research in Lie algebra theory and algebraic geometry, with applications in representation theory and beyond. By providing computational algorithms and implementations, we aim to facilitate further exploration and applications of these techniques in diverse mathematical contexts.

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