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Strange Quark Matter FRW Cosmological Model in Saez-Ballester Theory of Gravitation

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Abstract **Review Article**

This research work focuses on exploring a five-dimensional FRW cosmological model containing strange quark matter within the framework of the scalar-tensor theory of gravitation developed by Saez and Ballester. Additionally, various physical and kinematical parameters are calculated and analyzed. This work also includes an examination of the energy conditions related to the model. Furthermore, key cosmological parameters such as look-back time, luminosity distance, angular diameter distance, the age of the universe, and the particle horizon are analyzed.

Keywords: Five-dimensional FRW space-time, Strange Quark Matter, Saez-Ballester theory.

1. INTRODUCTION

Einstein's general theory of relativity has proven effective in developing various cosmological models of the universe. However, in recent years, several modifications to Einstein's gravitational theory have been proposed to address certain limitations of the original framework. For instance, Mach's principle is not fully embedded in Einstein's field equations. Additionally, the general theory of relativity does not adequately explain the early inflationary phase or the late-time accelerated expansion of the universe. To overcome these shortcomings, various extensions and modifications of general relativity have been introduced.

Saez and Ballester [1] formulated a scalar-tensor theory where the metric is coupled to a dimensionless scalar field ϕ in a straightforward way. Despite the scalar field being dimensionless, this theory allows for the emergence of an anti-gravity regime. Moreover, it provides a satisfactory explanation for weak gravitational fields and offers a potential solution to the 'missing matter' issue in non-flat Friedmann–Robertson–Walker (FRW) cosmological models. The field equations proposed by Saez and Ballester describe the interaction between the scalar and tensor fields.

$$R_{ij}-\frac{1}{2}g_{ij}R-\omega\phi^{n}\left(\phi_{,i}\,\phi_{,j}-\frac{1}{2}g_{ij}\phi_{,k}\,\phi^{,k}\right)=-8\pi T_{ij} \qquad (1)$$

And scalar field satisfies the equation

$$2\phi^{n}\phi^{i}_{;i}+n\phi^{n-1}\phi_{,k}\,\phi^{,k}=0 \qquad (2)$$

Where, $G_{ij}=R_{ij}-\frac{1}{2}Rg_{ij}$ is the Einstein tensor, R_{ij} is the Ricci tensor, R is the scalar curvature, n is arbitrary constant, ω is dimensionless coupling constant and T_{ij} is matter energy-momentum tensor. Here comma and semicolon denote partial and covariant differentiation respectively.

The exploration of cosmological models within the framework of scalar–tensor theories, particularly the Saez-Ballester theory, has remained a prominent area of research over the past few decades. Natenskar [2] investigated a five-dimensional Bianchi-type cosmological model involving strange quark matter under the Saez-Ballester framework. Other researchers, including Mahanta [3], Samanta [4], and Kumwat [5], have studied this theory in diverse gravitational contexts. Daimary [6, 7] examined cosmological models featuring bulk viscous strings in both five-dimensional spacetimes and five-dimensional FRW models under Saez-Ballester scalar-tensor theory. Tuan Do [8] focused on anisotropic power-law inflation within this theoretical framework. Additionally, several other researchers [9–14] have analyzed the Saez-Ballester theory in various gravitational scenarios.

This research paper is organized into five main sections. Section 2 presents the space-time structure and the corresponding field equations based on the scalar-tensor theory developed by Saez and Ballester, setting the foundation for the model. Section 3 focuses on deriving cosmological solutions by solving the field equations under appropriate assumptions. In Section 4, the physical and kinematical properties of the model are analyzed in detail, including the evaluation of parameters such as Hubble parameter, expansion scalar and shear scalar. Section 5 examines cosmological quantities like look-back time, luminosity distance, angular diameter distance, age of the universe, and the particle horizon. Finally, Section 6 provides a summary of the findings and conclusions drawn from the study.

2. METRIC AND FIELD EQUATIONS

The five-dimensional FRW space-time is given by,

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(1-kr^2\right)d\phi^2\right] \quad (3)$$

Non vanishing components of the Einstein tensor for the metric are

$$G_0^0 = \frac{6\dot{a}^2}{a^2} + \frac{6k}{a^2} \quad (4)$$

$$G_1^1 = G_2^2 = G_3^3 = G_4^4 = \frac{3\ddot{a}}{a} + \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \quad (5)$$

Where, an overhead dot indicates ordinary differentiation with respect to t and $k = +1, -1, 0$ for closed, open and flat models respectively.

Within the framework of this model, we have considered quark matter consists of massless up and down quarks, massive strange quarks, and electrons. In a simplified form of the bag model, quarks are considered to be non-interacting and massless.

Therefore, the quark pressure is given as,

$$p_q = \frac{\rho_q}{3}$$

Where, ρ_q is the quark energy density.

The total energy density is

$$\rho = \rho_q + B_c$$

The total pressure is

$$p = p_q - B_c$$

Now, the energy-momentum tensor representing strange quark matter is expressed as follows.

$$T_{ij} = \rho u_i u_j - \rho_s x_i x_j \quad (6)$$

Here ρ is the rest energy density for the cloud of strings with particles attached to them and ρ_s is the string tension density. They are related as, $\rho = \rho_p + \rho_s$, where, ρ_p is the particle energy density.

Therefore, instead of using the particle energy density in the string cloud, we consider the energy density of quark matter. As a result, we obtain,

$$\rho = \rho_q + \rho_s + B_c$$

Thus, based on equation (6), the energy-momentum tensor for strange quark matter associated with the string cloud can be expressed as,

$$T_{ij} = \left(\rho_q + \rho_s + B_c\right)u_i u_j - \rho_s x_i x_j \quad (7)$$

Here, u_i denotes the four-velocity of the particles, and x_i is the unit spacelike vector indicating the direction of the string. These vectors satisfy the conditions,

$$u_i u^i = -x_i x^i = 1$$

We have taken the direction of string along x-axes. Then the components of energy momentum tensor are

$$T_0^0 = \rho,\ T_1^1 = \rho_s,\ T_2^2 = T_3^3 = T_4^4 = 0 \qquad (8)$$

Now, from equations (1) to (5) and (8) the field equations for the metric can be written as

$$\frac{6\dot{a}^2}{a^2} + \frac{6k}{a^2} - \frac{\omega}{2}\phi^n \dot{\phi}^2 = -8\pi\rho \qquad (9)$$

$$\frac{3\ddot{a}}{a} + \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \frac{\omega}{2}\phi^n \dot{\phi}^2 = -8\pi\rho_s \qquad (10)$$

$$\frac{3\ddot{a}}{a} + \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \frac{\omega}{2}\phi^n \dot{\phi}^2 = 0 \qquad (11)$$

$$\frac{\ddot{\phi}}{\phi} + 4\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{n}{2}\frac{\dot{\phi}^2}{\phi^2} = 0 \qquad (12)$$

3. COSMOLOGICAL SOLUTION

The above equations (9) to (12) represents a system of three nonlinear differential equations involving five unknown variables. To obtain a determinate solution for this system, we adopt the power-law relation, which has been previously utilized [37]. This power-law relation connects the average scale factor $a(t)$ with the scalar field ϕ, and is expressed as,

$\phi \propto a^{k_1}$ where, $k_1 \geq 0$ is any integer. (13)

$$\phi = k_2 a^{k_1}$$

Without loss of generality, we take $k_2 = 1$, hence we get,

$$\phi = a^k \qquad (14)$$

Using equation (14) in field equation (9) – (12), the average scale factor is obtained in the form as,

$$a(t) = (k_4 t + k_5)^{\frac{1}{k_3+1}}$$

Where, $k_3 = 2 + \frac{n+1}{2}k$, k_4, k_5 are integrating constants.

$$a(t) = (k_4 t + k_5)^m \qquad (15)$$

Where, $m = \frac{1}{k_3 + 1}$.

The scalar field is calculated to be

$$\phi = (k_4 t + k_5)^{mk_1} \qquad (16)$$

Hence, the cosmological model in equation (3) for strange quark matter within the framework Saez-Ballester theory takes the form,

$$ds^2 = dt^2 - (k_4 t + k_5)^{2m}\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(1-kr^2\right)d\phi^2\right] \qquad (17)$$

From equation (9), (10) and (15), we can calculate the following parameters, The string energy density

$$\rho_s = k_6 (k_4 t + k_5)^{-(n-1)mk} - \frac{k_7}{(k_4 t + k_5)^2} \quad (18)$$

Where, $k_6 = \frac{\omega m k_1 k_4 k_7}{16\pi} = \frac{3}{8\pi}\left(3{k_4}^2 m^2 + m + 3k - 1\right)$

The string tension density

$$\rho = k_6 (k_4 t + k_5)^{-(n-1)mk} - \frac{6{k_4}^2 m^2}{(k_4 t + k_5)^2} - \frac{6k}{(k_4 t + k_5)^{2m}} \quad (19)$$

The string particle density is given by

$$\rho_p = \rho - \rho_s = \frac{k_7}{(k_4 t + k_5)^2} - \frac{6{k_4}^2 m^2}{(k_4 t + k_5)^2} - \frac{6k}{(k_4 t + k_5)^{2m}} \quad (20)$$

The quark energy density is given by

$$\rho_q = \rho - \rho_s - B_c = \frac{k_7}{(k_4 t + k_5)^2} - \frac{6{k_4}^2 m^2}{(k_4 t + k_5)^2} - \frac{6k}{(k_4 t + k_5)^{2m}} - B_c \quad (21)$$

The quark pressure is given by

$$p_q = \frac{\rho_q}{3} = \frac{k_7}{3(k_4 t + k_5)^2} - \frac{6{k_4}^2 m^2}{3(k_4 t + k_5)^2} - \frac{6k}{3(k_4 t + k_5)^{2m}} - \frac{B_c}{3} \quad (22)$$

4. Physical and Kinematical Properties

A detailed analysis has been carried out on several physical and kinematical parameters associated with the derived model. These parameters provide crucial insights into the dynamic behavior and physical viability of the cosmological model under investigation. The study includes, but is not limited to, the examination of quantities such as the expansion scalar, shear scalar, Hubble parameter and deceleration parameter. Each of these parameters plays a significant role in understanding the evolution, geometry, and physical characteristics of the universe.

Spatial Volume $V = (k_4 t + k_5)^{3m}$ (23)

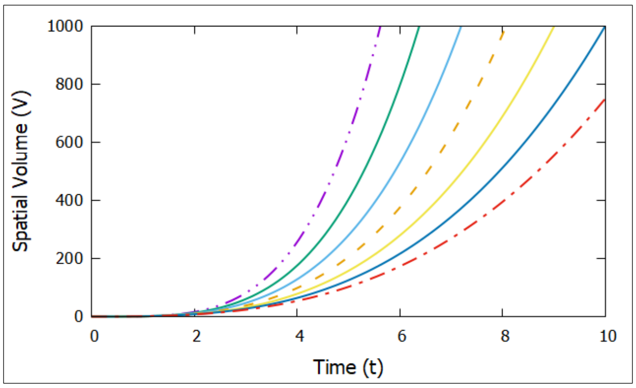

Figure 1- Plot of Spatial Volume vs. Cosmic time for $k_4 = 1, k_5 = 0$ and various values of m

In figure 1 spatial volume is plotted against cosmic time. As time progresses, the spatial volume of the universe increase without bound, indicating that the universe is undergoing continuous and unending expansion. This behaviour implies that the scale factor [as in equation (15)] grows over time, leading to an ever-expanding geometry. The infinite growth of spatial volume reflects the dynamic nature of cosmic evolution. Such a trend is consistent with observations of the accelerating universe

Hubble Parameter $H = \frac{m k_4}{(k_4 t + k_5)}$ (24)

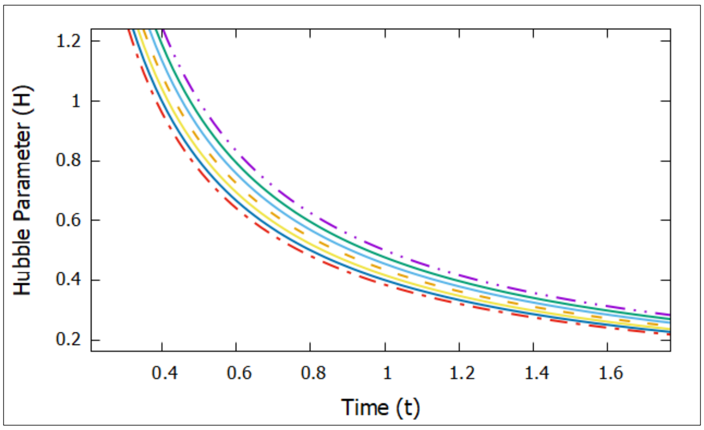

Figure 2- Plot of the Hubble Parameter vs. Cosmic time for $k_4 = 1, k_5 = 0$ and the various values of m

Expansion Scalar $\theta = 3H = \dfrac{3k_4 m}{(k_4 t + k_5)}$ (25)

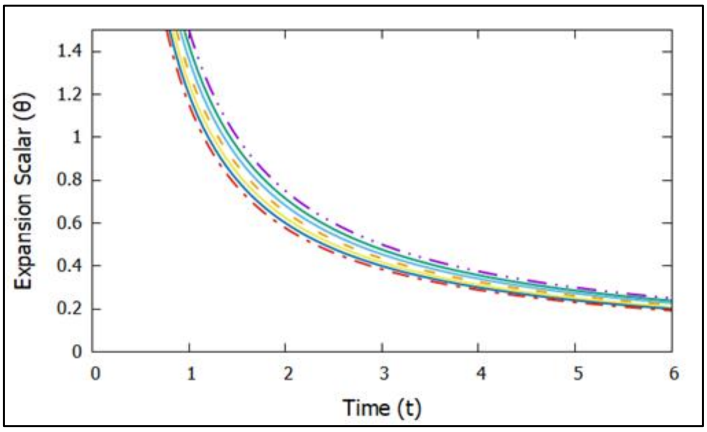

Figure 3- Plot of the Expansion Scalar vs. Cosmic time for $k_4 = 1, k_5 = 0$ and the various values of m

The Hubble parameter and the expansion scalar are graphed over time in figure 2 and figure 3, and both display a declining pattern. This reduction indicates that the universe is undergoing ongoing expansion. In particular, the decreasing expansion scalar suggests that the rate of expansion is accelerating.

Shear Scalar $\sigma^2 = \dfrac{1}{2}\sigma_{ij}\sigma^{ij} = \dfrac{3}{2}\dfrac{{k_4}^2 m^2}{(k_4 t + k_5)^2}$ (26)

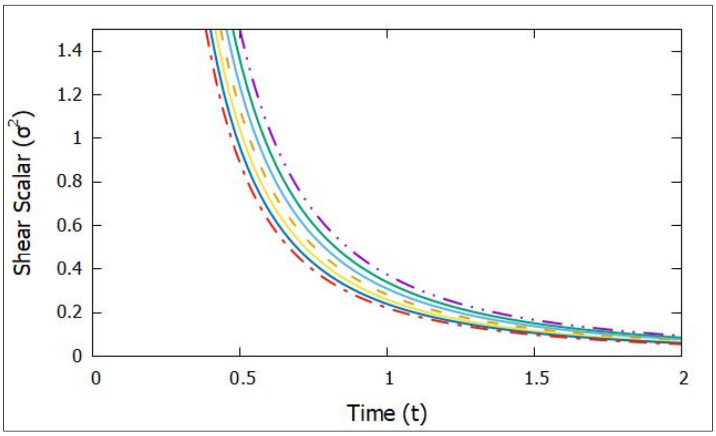

Figure 4- Plot of Shear Scalar vs. Cosmic time for $k_4 = 1, k_5 = 0$ and various values of m

Figure 4 illustrates the shear scalar as a decreasing function of time. Although the shear scalar diminishes over time, it never fully reduces to zero, indicating that the anisotropic nature of the universe weakens gradually but never entirely disappears. Additionally, since $\lim_{t\to\infty}\frac{\sigma}{\theta}\neq 0$, the universe does not attain perfect isotropy even at late times. Therefore, the resulting cosmological model remains shearing and non-rotating.

Deceleration parameter $q=\frac{-a\ddot{a}}{\dot{a}^2}, q=\frac{1}{m}-1$ (27)

If $m>1 \Rightarrow q<0$: accelerating universe.

If $m=1 \Rightarrow q=0$: uniform expansion of the universe.

If $0<m<1 \Rightarrow q<0$: decelerating universe.

5. Kinematic Tests

Various kinematic tests are examined in this section to analyze the dynamic behavior and evolution of the Universe within the framework of the proposed cosmological model.

Redshift-

In the context of redshift, the average scale factor is connected to the redshift through the relation,

Where, subscript 0 denotes the present phase and a_0 is the present scale factor

$$1+z=\frac{a_0}{a}$$

Hence, we get,

$$1+z=\left(\frac{t_0}{t}\right)^m \qquad (28)$$

Here, t_0 is the age of universe at present time.

$$t=t_0(1+z)^{-\frac{1}{m}} \qquad (29)$$

Look Back Time-

Look-back time represents the interval between the current age of the universe and the age it had when the light from a distant celestial object was originally emitted. In essence, it indicates how far into the past we are looking when we observe faraway galaxies or other cosmic phenomena. Mathematically look back time is given as, [15, 16]

$$t_L=t_0-t(z) \qquad (30)$$

By solving equations (24), (29) and (30), we get

$$t_L=\frac{m}{H_0}\left[1-\frac{1}{(1+z)^{\frac{1}{m}}}\right] \qquad (31)$$

Here, H_0 represents the Hubble constant at the present time $(t=0)$ and its value typically falls within the range of $50 \text{ to } 100\, km/\sec/Mpc$.

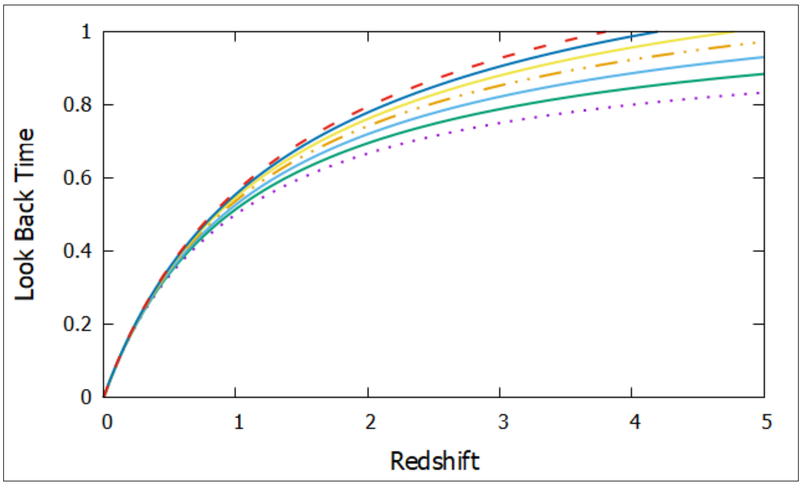

Figure 5- Plot of Look Back Time vs. Cosmic time for various values of m

Luminosity Distance-

The **luminosity distance** d_L of a light source is described based on the relationship between its **intrinsic luminosity** L (the total amount of energy emitted per unit time) and the **observed energy flux** l (the amount of energy received per unit area per unit time by an observer) i.e., $d_L{}^2=\frac{L}{4\pi l}$

Due to redshift and time dilation, the cosmological luminosity distance is modified. It becomes,

$$d_L=a_0\, r_1(z)(1+z) \qquad (32)$$

The radial coordinate distance $r_1(z)$ of the object at the time of light emission is given by the following expression.

$$r_1(z)=\int_t^{t_0}\frac{dt}{a}$$

$$r_1(z)=\frac{mH_0^{-1}a_0^{-1}}{1-m}\left[1-(1+z)^{-\left(\frac{1}{m}-1\right)}\right] \qquad (33)$$

Now, from equation (32) and (33), the Luminosity distance is calculated to be

$$d_L=\frac{m}{(1-m)H_0}\left[1-(1+z)^{-\left(\frac{1}{m}-1\right)}\right](1+z) \qquad (34)$$

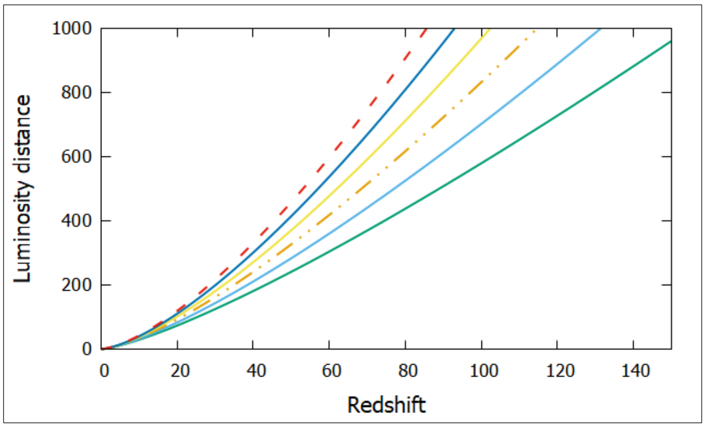

Figure 6- Plot of Luminosity Distance vs. Cosmic time for various values of m

Angular Diameter Distance-

The **angular diameter distance** d_A is a cosmological measure that relates the **physical size** of an object to its **angular size** (as seen by an observer), defined mathematically as,

$$d_A=\frac{d_L}{(1+z)^2}$$

From equation (49), we get, the angular diameter distance for the obtained cosmological model,

$$d_A=\frac{m}{(1-m)H_0}\frac{1-(1+z)^{-\left(\frac{1}{m}-1\right)}}{(1+z)} \qquad (35)$$

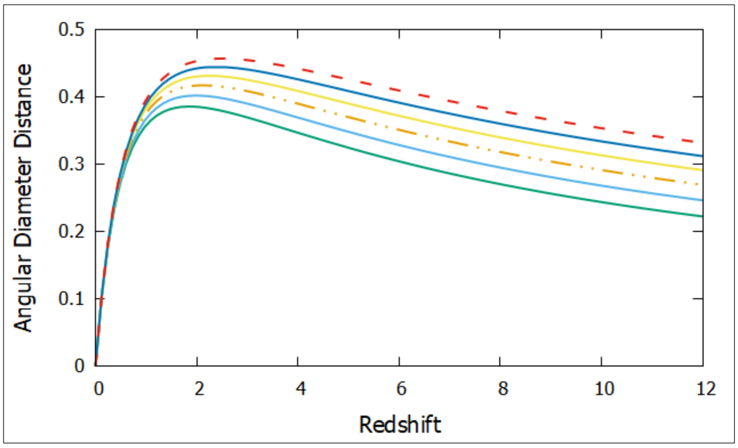

Figure 7- Plot of Angular Diameter Distance vs. Cosmic time for various values of m

Age of the universe-

The current age of the universe is determined using the following approach. [17, 18]

$$\text{Age of Universe}=\int_0^t dt=\int_0^a\frac{da}{aH}=\int_0^z\frac{dz}{(1+z)H}$$

Which gives

$$\text{Age of Universe}=\frac{1}{mH_0}\left[1-(1+z)^{-\frac{1}{m}}\right] \qquad (36)$$

Particle Horizon-

Since the Universe has a finite age and the speed of light is limited, there exists a particle horizon, denoted by H_p. This horizon marks the greatest distance from which light or any information could have reached us since the beginning of the Universe, effectively defining the boundary of the observable universe. The expression for the particle horizon corresponding to the derived model is given by, [19]

$$H_p = \lim_{t_p \to 0} a_0 \int_0^z \frac{dt}{a(t)} = \lim_{z \to \infty} \int_0^z \frac{dz}{H(z)} \qquad (37)$$

Here, t_p represents the moment in the past when the light signal was originally emitted by the source. From equation (37), we get,

$$H_p = \lim_{z \to \infty} \int_0^z \frac{t_0}{m(1+z)^{\frac{1}{m}}} dz$$

The particle horizon in the proposed model is calculated by performing the integration of the previously stated equation.

$$H_p = \frac{m}{(1-m)H_0} \qquad (38)$$

6. CONCLUSION

In this study, a five-dimensional FRW cosmological model has been examined within the framework of Saez-Ballester theory of gravitation, considering Strange Quark Matter as the gravitational source. The resulting model exhibits an expanding universe. The calculated deceleration parameter indicates that this expansion is accelerating. Additionally, various physical characteristics of the model have been analyzed to understand its dynamical behaviour.

The analysis of the five-dimensional FRW cosmological model in the framework of Saez-Ballester theory with Strange Quark Matter reveals several key features of cosmic evolution. The model describes a continuously expanding universe, as evidenced by the unbounded increase in spatial volume over time. The declining trends of the Hubble parameter and expansion scalar further confirm this expansion, with the latter suggesting an accelerated rate. The behavior of the shear scalar, although decreasing, remains non-zero, indicating that while the universe becomes less anisotropic over time, it never reaches complete isotropy. Consequently, the model portrays a universe that evolves dynamically, remains shearing and non-rotating, and aligns well with current observational evidence of accelerated cosmic expansion.

The cosmological analysis has been extended to include essential observational parameters such as redshift, look-back time, luminosity distance, angular diameter distance, age of the universe, and particle horizon. The look-back time and luminosity distance are found to increase with redshift, which aligns with standard cosmological expectations, indicating that we observe older and more distant objects at higher redshifts. Interestingly, the angular diameter distance first increases with redshift and then begins to decrease beyond a certain point. This non-monotonic behavior reflects the geometry of the expanding universe, where extremely distant objects appear larger on the sky due to light being emitted when the universe was more compact. These trends are consistent with observational data and support the model's physical validity. The particle horizon illustrates the finite region of the observable universe due to the limits of light travel time. Overall, these findings reinforce the physical plausibility of the proposed cosmological model and its agreement with the observed features of our accelerating universe.

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