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A Four-Step Iteration Scheme to Approximate Fixed Points of Generalized (α, β)-Nonexpansive Type 1 Mappings with Application

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Abstract **Original Research Article**

In this paper, we introduce a four-step iteration scheme for approximating fixed points of generalized (α, β)-nonexpansive type 1 mapping. We prove the convergence and stability results for contractive-like mapping in uniformly convex Banach space. We also prove weak and strong convergence results for generalized (α, β)-nonexpansive type 1 mapping. We provide numerical examples to show that this scheme is faster than Agarwal, Abbas, Thakur, Ullah and S^{**} -iteration scheme. We used this iteration method to find the solution of a nonlinear integral equation.
Keywords: Uniformly Convex Banach space, Contractive like mapping, Generalized (α, β)- nonexpansive type1 mapping, Stability, Weak and Strong Convergence Theorem, Nonlinear Integral equation.
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1. INTRODUCTION

Let X be a Banach space and C be any nonempty subset. A mapping T : C → C is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, y ∈ C. In 1965, Browder[7] and Gohde[5] independently proved that "Every nonexpansive self-mapping of a closed convex and bounded subset of uniformly convex Banach space has a fixed point". The concept of generalized (α, β)-nonexpansive type 1 mapping was introduced by Akutsah et al. [6] in 2021. A mapping T: C → C is said to be generalized (α, β) -nonexpansive type 1 if there exist α, β, λ ∈ [0, 1) with α ≤ β and α + β < 1 such that for all x, y ∈ C, λ‖T x − x‖ ≤ ‖x − y‖, then

$$\|Tx - Ty\| \leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|.$$

Since every nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping, but converse need not to be true. For example

Example 1.1. [6] Let C = {(0, 0), (1, 0), (3, 0)} be a subset of R^2 with norm ‖.‖ on C defined by

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Then (C, ‖.‖) is an Banach space. Define a mapping T: C → C by

$$T(x) = \begin{cases} (0,0) \, if \; x \in \{(0,0),(1,0)\} \\ (1,0) \, if \; x \in (3,0). \end{cases}$$

Here T is generalized (α, β)-nonexpansvie type 1 mapping with λ = $\frac{1}{10}$, α = $\frac{1}{2}$ and β = $\frac{1}{3}$, but T is not nonexpansive mapping.

Fixed point theory provides very useful tools to solve most of the nonlinear problems, that have application in different fields, as they can be easily transformed into a fixed point problem. Fixed point problems possess either an existing solution or an approximate solution. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. Over the years researchers have developed many iterative schemes to obtain approximate solutions of fixed point problems for different mappings or operators over different spaces. The iterative method has two main aspects fastness and stability. In 1953, Mann [18] introduced a new iterative scheme to approximate the fixed points of nonexpansive mappings. In 1974, Ishikawa [15] introduced a two-step Mann iterative scheme to approximate fixed points of pseudo-contractive mappings. Many authors studied Mann and Ishikawa's iterative schemes for the approximation of fixed points of nonexpansive mappings. In 2000, Noor [13] introduced a new iterative scheme. In 2007, Agarwal *et al.* [14] introduced a two-step iteration process for nearly asymptotically nonexpansive mappings. They prove that this iteration process converges faster than the Mann iteration process. In 2014, Abbas and Nazir [12] developed an iterative scheme that is faster than Agarwal et al. [14] scheme for contraction mappings. In 2016, Thakur *et al.*[3] developed an iterative procedure and prove that this iteration process is faster than Abbas and Nazir [12]. In 2018, Ullah and Arshad [11] developed a new iteration process that converges faster than all the aforementioned processes. In 2023 Rawat *et al.*[16] introduced a new iteration process is called S^{**} -iteration process, as follows: For each $x_0 \in$ C, the sequence $\{x_n\}$ in C is defined by

$$\left\{\begin{aligned} x_{n+1} &= T((1-\alpha_n)Tz_n + \alpha_n Ty_n) \\ y_n &= T((1-\beta_n)z_n + \beta_n Tz_n) \\ z_n &= T((1-\gamma_n)x_n + \gamma_n Tx_n). \end{aligned}\right. \qquad (1.1)$$

For all n ≥ 0 and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in (0, 1). They proved that this iteration process is faster than Agarwal, Abbas, Thakur, and Ullah iteration schemes. In this paper, we introduce a new four-step iteration process which is faster than Agarwal, Abbas, Thakur, Ullah, and S^{**} iteration processes with the help of numerical example and prove the convergence results using our iterative scheme for generalized (α, β)- nonexpansive type 1 mapping in uniformly convex Banach spaces. We also prove that our process is stable.

2 Preliminaries

Definition 2.1. [10] A Banach space X is said to be uniformly convex if for each ϵ ∈ (0, 2] there is a δ > 0 such that x, y ∈ X

$$\|(x+y)/2\| \le 1-\delta \text{whenever} \|x-y\| \ge \epsilon \text{ and } \|x\| = \|y\| = 1.$$

Definition 2.2. [20]A Banach space X is said to satisfy Opial's property if for each sequence $\{x_n\}$ in X converging weakly to x ∈ X, we have

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|,$$

for all x, y ∈ X s.t. x ≠ y.

Definition 2.3. [4] Let X be a Banach space and let T : X → X be a self map. The mapping T is called contractive like mapping if there exist a constant δ ∈ [0, 1) and a strictly increasing and continuous function ξ : [0, ∞) → [0, ∞) with ξ(0) = 0 such that for all x, y ∈ X,

$$\|Tx - Ty\| \le \delta\|x-y\| + \xi(\|x - Tx\|). \qquad (2.1)$$

Definition 2.4. Let X be a Banach space and C be any nonempty subset of X. Let T : C → C be said to be nonexpansive if for each x, y ∈ C

$$\|Tx - Ty\| \le \|x-y\|.$$

Definition 2.5. A mapping T : C → C is called quasi non-expansive if for all p ∈ C and q ∈ F (T), we have

$$\|Tp - q\| \le \|p-q\|.$$

Definition 2.6. [8]A mapping T : C → C is said to satisfy condition I, if ∃ a non decreasing function f : [0, ∞) → [0, ∞) with f (0) = 0 and f (c) > 0 for all c > 0 s.t. ‖x − T x‖ ≥ f (d(x_n , F (T))), for all x ∈ C, where d(x, F (T)) = inf{‖x − x^* ‖ : x^* ∈ F (T)}.

Definition 2.7. [6] Let C be a nonempty subset of a Banach space X. A mapping T : C → C is said to be generalized (α, β) -nonexpansive type 1 if there exist α, β, λ ∈ [0, 1) with α ≤ β and α + β < 1 such that for all x, y ∈ C, λ‖T x − x‖ ≤ ‖x − y‖ then

$$\|Tx - Ty\| \le \alpha\|y - Tx\| + \beta\|x - Ty\| + (1-(\alpha+\beta))\|x-y\|$$

Remark 2.1. [6] It is easy to see that the following statements are true ;

1. If α = β = 0 and λ = $\frac{1}{2}$, than the generalized (α, β)-nonexpansive type 1 mapping satisfying condition (C).
2. If α = β = 0 and λ ∈ [0, 1), than the generalized (α, β)-nonexpansive type 1 mapping satisfying condition (C_λ).

Proposition 2.1. [6] Let C be a nonempty subset of a Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ. Then T is quasi-nonexapansive.

Proposition 2.2. [6] Let C be a nonempty subset of a Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping. Then F (T) is closed. Furthermore, if X is strictly convex and C is convex, then F (T) is convex.

Theorem 2.3. [6] Let C be a nonempty closed subset of a Banach space X with Opial property and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with λ = $\frac{1}{2}$ γ ∈ [0, 1). If $\{x_n\}$converges weakly to x and $\lim_{n\to\infty} \| T x_n - x_n \| = 0$ then. T x = x That is I − T is demiclosed at zero, where I is the identity mapping on X.

Theorem 2.4. [6] Let C be a nonempty compact subset of a Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with λ = $\frac{\gamma}{2}$ γ ∈ [0, 1). Then T has a fixed point in C if and only if T admits an almost fixed point sequence.

Definition 2.8. [1] Let $\{t_n\}$ be any arbitrary sequence in X. Then an iteration procedure x_{n+1} = f (T, x_n), converging to fixed point p, is said to T -stable, if for $\epsilon_n = \| t_{n+1} - f(T, t_n)\|$, ∀n ∈ N , we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n$ = p.

Lemma 2.5. [17] If λ is a real number such that 0 ≤ λ < 1 and $\{\epsilon_n\}$ is the sequence of positive numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0$$

then for a sequence of positive numbers v_n satisfying

$$v_{n+1} \le \lambda v_n + \epsilon_n, \text{for} n = 1,2,\ldots,$$

we have

$$\lim_{n\to\infty} v_n = 0.$$

Lemma 2.6. [9]Let X be a uniformly convex Banach space and 0 < p ≤ t_n ≤ q < 1 ∀n ∈ N . Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X s.t. $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a$ hold for some a ≥ 0. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proposition 2.7. [6]All mappings satisfying condition (C_λ) is a generalized (α, β)- nonexpansive type 1 mapping.

3 New Iteration Process

We introduce a new iteration process for finding fixed points in Banach space, as follows: For each x_0 ∈ C, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} w_n = T((1-\gamma_n)x_n + \gamma_n T x_n) \\ z_n = T((1-\beta_n)w_n + \beta_n T w_n) \\ y_n = T((1-\alpha_n)T w_n + \alpha_n T z_n) \\ x_{n+1} = T y_n. \end{cases} \quad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ ∈ (0, 1).

4 Convergence Theorems for Generalized (α, β)-Nonexpansive Type 1 Mappings

Lemma 4.1. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ. Suppose that $\{x_n\}$ is defined by (3.1). Then we have the following:

1. $\{x_n\}$ is bounded.
2. $\lim_{n\to\infty} \|T x_n - p\|$ exists for all p ∈ F (T).

Proof. Let p ∈ F (T). Using iteration (3.1) and Proposition (2.1), we obtain

$$\begin{aligned} \|w_n - p\| &= \|T((1-\gamma_n)x_n + \gamma_n T x_n) - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \quad (4.1) \end{aligned}$$

$$\begin{aligned} \|z_n - p\| &= \|T((1-\beta_n)w_n + \beta_n T w_n) - p\| \\ &\le (1-\beta_n)\|w_n - p\| + \beta_n\|Tw_n - p\| \\ &\le (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\| \\ &= \|w_n - p\|. \quad (4.2) \end{aligned}$$

From equation (4.1) and (4.2), we get

$$\|z_n - p\| \le \|x_n - p\|. \quad (4.3)$$

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n)Tw_n + \alpha_n T z_n) - p\| \\ &\le (1-\alpha_n)\|Tw_n - p\| + \alpha_n\|Tz_n - p\| \\ &\le (1-\alpha_n)\|w_n - p\| + \alpha_n\|z_n - p\|. \quad (4.4) \end{aligned}$$

From equation (4.1), (4.3) and (4.4) we get,

$$\|y_n - p\| \leq \|x_n - p\|. \qquad (4.5)$$

Now

$$\|x_{n+1} - p\| = \|Ty_n - p\|$$
$$\leq \|y_n - p\|. \qquad (4.6)$$

From equation (4.5) and (4.6), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\|. \qquad (4.7)$$

This shows that $\{\|x_n - p\|\}$ is bounded and non-increasing $\forall p \in F(T)$. Thus $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists.

Lemma 4.2. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ Suppose that $\{x_n\}$ is defined by (3.1). Then $\lim_{n\to\infty} \|T x_n - x_n\| = 0$.

Proof. Suppose that F (T) ≠ ϕ and let p ∈ F (T). By Lemma 4.1 $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Let

$$\lim_{n\to\infty} \|x_n - p\| = a. \qquad (4.8)$$

Case I: a = 0, we are done. Case II: a > 0, From equation (4.1) in Lemma 4.1, we have

$$\|w_n - p\| \leq \|x_n - p\|.$$

Taking lim sup of both sides, we have

$$\limsup_{n\to\infty} \|w_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = a \qquad (4.9)$$

Now

$$\|Tx_n - p\| \leq \|x_n - p\|$$
$$\Rightarrow \limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = a. \qquad (4.10)$$

On the other hand

$$\|x_{n+1} - p\| = \|Ty_n - p\|$$
$$\leq \|y_n - p\|$$
$$= \|T((1-\alpha_n)Tw_n + \alpha_n Tz_n) - p\|$$
$$\leq (1-\alpha_n)\|Tw_n - p\| + \alpha_n\|Tz_n - p\|$$
$$\leq (1-\alpha_n)\|w_n - p\| + \alpha_n\|z_n - p\|$$
$$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|w_n - p\|.$$

This implies that

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \leq \|w_n - p\| - \|x_n - p\|$$
$$\|x_{n+1} - p\| - \|x_n - p\| \leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n}$$
$$\leq \|w_n - p\| - \|x_n - p\|.$$

Therefore

$$\|x_{n+1} - p\| \leq \|w_n - p\|$$
$$a \leq \liminf_{n\to\infty} \|w_n - p\|. \qquad (4.11)$$

From equation (4.9) and (4.11), we get

$$a = \lim_{n\to\infty} \|w_n - p\|$$
$$= \lim_{n\to\infty} \|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\|$$
$$= \lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\|. \qquad (4.12)$$

From (4.8), (4.10), (4.12), and Lemma 2.6, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Theorem 4.3. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ. Suppose that $\{x_n\}$ is defined by (3.1). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. In Lemma 4.1, we have established $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in F (T). Let u and v be weak limits of the sequence $\{x_{n_i}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ respectively. By Lemma 4.2, $\lim_{n\to\infty} \|x_n - T x_n\| = 0$ and I − T is demi closed with respect to zero by Theorem 2.3, we, therefore, have that T u = u. Using a similar approach, we can show that T v = v. In what follows, we establish uniqueness. From Lemma 4.1, we have that $\lim_{n\to\infty} \|x_n - u\|$ exists. Now, suppose that u ≠ v then by Opial's condition

$$\begin{aligned}\lim_{n\to\infty}\|x_n-u\| &= \lim_{k\to\infty}\|x_{n_k}-u\|\\ &< \lim_{k\to\infty}\|x_{n_k}-v\|\\ &= \lim_{n\to\infty}\|x_n-v\|\\ &= \lim_{i\to\infty}\|x_{n_i}-v\|\\ &< \lim_{i\to\infty}\|x_{n_i}-u\|\\ &= \lim_{n\to\infty}\|x_n-u\|\end{aligned}$$

This is a contradiction, so u = v. Hence, $\{x_n\}$ is converges weakly to a fixed point of F (T) and this completes the proof.

Theorem 4.4. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ Suppose that $\{x_n\}$ is defined by (3.1). Then $\{x_n\}$ converges strongly to fixed of F (T) if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

where d(x, F (T)) = inf{‖x − p‖ : p ∈ F (T)}.

Proof. Let $\{x_n\}$ converges to p, then $\lim_{n\to\infty}$ d(x_n , p) = 0.It follows that $\lim_{n\to\infty}$ d(x_n , F (T)) = 0. Therefore, lim $\inf_{n\to\infty}$ d(x_n , F (T)) = 0.

Conversely: Suppose that $\lim_{n\to\infty}$ d(x_n, F (T)) = 0. It follows that from Lemma 4.1, we have $\lim_{n\to\infty}$ $\|x_n - x^*\|$ exists and that lim $\inf_{n\to\infty}$ d(x_n, F (T)) exists for all p ∈ F (T). Our assumption lim $\inf_{n\to\infty}$ d(x_n, F (T)) = 0, therefore we have $\lim_{n\to\infty}$ d(x_n, F (T)) = 0. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence in F (T), s.t.

$$\|x_{n_k}-u_k\| < \frac{1}{2^k}$$

It follows from (4.7) that $\|x_{n+1}-u_k\| \le \|x_n-u_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned}\|u_{k+1}-u_k\| &\le \|u_{k+1}-x_{n+1}\| + \|x_{n+1}-u_k\|\\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k}\\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus we have that $\{u_k\}$ is a cauchy sequence in F (T). Also by Proposition 2.2, we have that F (T) is closed. Thus $\{u_k\}$ is convergent sequence in F (T). Now suppose that $\{u_k\}$ converges to p ∈ F (T). Therefore since

$$\|x_{n_k}-p\| \le \|x_{n_k}-u_k\| + \|u_k-p\| \to 0, \text{ as } k\to\infty.$$

as k → ∞. We obtain that $\lim_{k\to\infty}$ $\|x_{n_k} - p\|$ = 0 and so $\{x_{n_k}\}$ converges strongly to p ∈ f (T). Since $\lim_{n\to\infty}$ $\|x_n - p\|$ exists. It follows that $\{x_n\}$ converges strongly to p.

Theorem 4.5. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T : C → C be a generalized (α, β)-nonexpansive type 1 mapping with F (T) ≠ ϕ Suppose that $\{x_n\}$ is defined by (3.1). Let T satisfy condition I. Then $\{x_n\}$ converges strongly to a fixed of T.

Proof. By Lemma 4.1, we have $\lim_{n\to\infty}$ $\|T x_n - p\|$ exists and Lemma 4.2, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Using the fact ∀ x ∈ C

$$\begin{aligned}0 &\le \lim_{n\to\infty} f(d(x_n, F(T)))\\ &\le \lim_{n\to\infty}\|x_n - Tx_n\|\\ &= 0\end{aligned}$$

and that

$$\lim_{n\to\infty} f(d(x_n, f(T))) = 0.$$

Since f is nondecreasing with f (0) and f (c) > 0 for c ∈ (0, ∞), it follows that $\lim_{n\to\infty}$ d(x_n , F (T)) = 0. Hence by Theorem 4.4, $\{x_n\}$ converges strongly to p ∈ F (T).

5 Rate of Convergence

In this section, we have to show that New-iteration is faster than $S^{\star\star}$ -iteration [16], Ullah[11], Thakur[3], Abbas[12] and Agarwal[14] with the help of Numerical examples

Example 5.1. Let X = R and C = [0, 50]. T : C → C be a mapping defined by $Tx = \sqrt{x^2 - 6x + 84}$ for all x ∈ C. Clearly T is contractive like mapping and its unique fixed point p = 14. For $\{\alpha_n\}$ = $\{\beta_n\}$ =$\{\gamma_n\}$= $\frac{2}{3}$ and initial value x_0 = 20. The comparison table 1 shows that the New- iteration (3.1) converges to x^* = 14 faster than $S^{\star\star}$ -iteration[16], Ullah[11], Thakur[3], Abbas[12] and Agarwal[14]. The convergence behavior of these iteration processes is represented in figure 1.

Table 1: Comparison of the rate of convergence with different iteration process

Iteration	New-iteration	S ⋆⋆-iteration	Ullah	Thakur	Abbas	Agarwal.
0	20.00000000	20.00000000	20.00000000	20.00000000	20.00000000	20.00000000
1	16.08034078	16.54907228	17.34936358	18.20009101	18.14028149	18.71536305
2	14.59661436	14.93901513	15.72486926	16.84141256	16.75334733	17.64435796
3	14.15775552	14.32037796	14.83838456	15.86617544	15.77371757	16.77299902
4	14.04068401	14.10594624	14.39367556	15.19708855	15.11444356	16.08067821
5	14.01042164	14.03465017	14.18153229	14.75462170	14.68764370	15.54252794
6	14.00266496	14.01129059	14.08297062	14.46996607	14.41909751	15.13221444
7	14.00068116	14.00367451	14.03776498	14.29033683	14.25338952	14.82442436
8	14.00017408	14.00119539	14.01715624	14.17843536	14.15243081	14.59657510
9	14.00004449	14.00038884	14.00778708	14.10930375	14.09141268	14.42965122
10	14.00001137	14.00012647	14.00353309	14.06681928	14.05471645	14.30833480
11	14.00000291	14.00004114	14.00160271	14.04079626	14.03271392	14.22069311
12	14.00000074	14.00001338	14.00072698	14.02488869	14.01954558	14.15766010
13	14.00000019	14.00000435	14.00032974	14.01517671	14.01167308	14.11247359
14	14.00000005	14.00000142	14.00014956	14.00925182	14.00696972	14.08015739
15	14.00000001	14.00000046	14.00006783	14.00563897	14.00416084	14.05708522
16	14.00000000	14.00000015	14.00003077	14.00343657	14.00248376	14.04063307
17	14.00000000	14.00000005	14.00001395	14.00209422	14.00148256	14.02891182
18	14.00000000	14.00000002	14.00000633	14.00127615	14.00088492	14.02056633
19	14.00000000	14.00000001	14.00000287	14.00077762	14.00052819	14.01462705
20	14.00000000	14.00000000	14.00000130	14.00047384	14.00031526	14.01040157
21	14.00000000	14.00000000	14.00000059	14.00028873	14.00018817	14.00739605
22	14.00000000	14.00000000	14.00000027	14.00017593	14.00011231	14.00525861
23	14.00000000	14.00000000	14.00000012	14.00010720	14.00006703	14.00373871
24	14.00000000	14.00000000	14.00000006	14.00006532	14.00004001	14.00265801
25	14.00000000	14.00000000	14.00000002	14.00003980	14.00002388	14.00188965
26	14.00000000	14.00000000	14.00000001	14.00002425	14.00001425	14.00134338
27	14.00000000	14.00000000	14.00000001	14.00001478	14.00000851	14.00095502
28	14.00000000	14.00000000	14.00000000	14.00000900	14.00000508	14.00067892
29	14.00000000	14.00000000	14.00000000	14.00000549	14.00000303	14.00048264
30	14.00000000	14.00000000	14.00000000	14.00000334	14.00000181	14.00034311

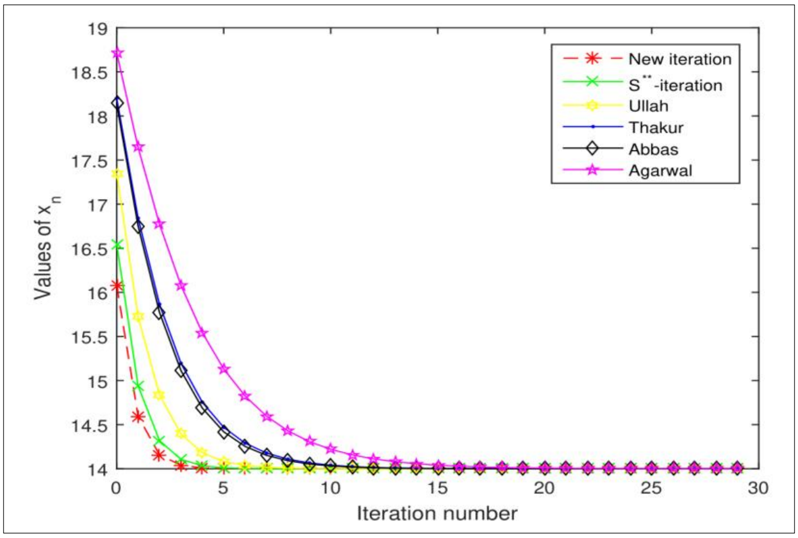

Figure 1: Graphical representation of convergence of iterative method

6 Stability Theorems

In this section, we have to prove that our iteration process is T -stable.

Theorem 6.1. Let C be a nonempty closed convex subset of a uniformly convex Banach spaces X. Let T be a mapping satisfying (2.1) and p be a fixed point of T. Let $\{x_n\}$ be defined by (3.1) with sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (0, 1). Then the iteration (3.1) is T − stable.

Proof. Let $\{x_n\}$ be an arbitrary sequence in X and the sequence is generated by (3.1) is t_{n+1} = f (T, t_n) converging to a unique fixed point p and $\epsilon_n = \|x_{n+1} - f(T, x_n)\|$. We show that $\lim_{n\to\infty} \epsilon_n = 0$ iff $\lim_{n\to\infty} x_n = p$.
Suppose that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \|x_{n+1}-f(T,x_n)\| + \|f(T,x_n)-p\| \\
&\leq \epsilon_n + \|Ty_n - p\| \\
&\leq \epsilon_n + \delta\|y_n - p\| \\
&= \epsilon_n + \delta\|T((1-\alpha_n)Tw_n + \alpha_n Tz_n) - p\| \\
&\leq \epsilon_n + \delta^2\|(1-\alpha_n)Tw_n + \alpha_n Tz_n - p\| \\
&\leq \epsilon_n + \delta^2[(1-\alpha_n)\|Tw_n - p\| + \alpha_n\|Tz_n - p\|] \\
&\leq \epsilon_n + \delta^3[(1-\alpha_n)\|w_n - p\| + \alpha_n\|z_n - p\|] \\
&= \epsilon_n + \delta^3[(1-\alpha_n)\|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\| \\
&\quad +\alpha_n\|T((1-\beta_n)w_n + \beta_n Tw_n) - p\|] \\
&\leq \epsilon_n + \delta^4[(1-\alpha_n)\|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\
&\quad +\alpha_n\|(1-\beta_n)w_n + \beta_n Tw_n - p\|] \\
&\leq \epsilon_n + \delta^4[(1-\alpha_n)[(1-\gamma_n)\|x_n - p\| + \gamma_n\delta\|x_n - p\|] \\
&\quad +\alpha_n[(1-\beta_n)\|w_n - p\| + \beta_n\delta\|w_n - p\|]] \\
&= \epsilon_n + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\|w_n - p\|] \\
&= \epsilon_n + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\|] \\
&\leq \epsilon_n + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\delta(1-\gamma_n(1-\delta))\|x_n - p\|] \\
&= \epsilon_n + \delta^4[((1-\alpha_n)(1-\gamma_n(1-\delta)) \\
&\quad +\alpha_n\delta(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta)))\|x_n - p\|]. \qquad (6.1)
\end{aligned}$$

Since δ ∈ [0, 1) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in (0, 1). We have

$$\delta^4((1-\alpha_n)(1-\gamma_n(1-\delta)) + \alpha_n\delta(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))) < 1.$$

Hence by Lemma 2.5, we have $\lim_{n\to\infty} \|x_n - p\| = 0$, which gives $\lim_{n\to\infty} x_n = p$.

Conversely: Suppose that $\lim_{n\to\infty} x_n = p$. Then

$$\begin{aligned}
\epsilon_n &= \|x_{n+1} - f(T,x_n)\| \\
&\leq \|x_{n+1}-p\| + \|f(T,x_n) - -p\| \\
&\leq \|x_{n+1}-p\| + \|Ty_n - p\| \\
&\leq \|x_{n+1}-p\| + \delta\|y_n - p\| \\
&= \|x_{n+1}-p\| + \delta\|T((1-\alpha_n)Tw_n + \alpha_n Tz_n) - p\| \\
&\leq \|x_{n+1}-p\| + \delta^2\|(1-\alpha_n)Tw_n + \alpha_n Tz_n - p\| \\
&\leq \|x_{n+1}-p\| + \delta^2[(1-\alpha_n)\|Tw_n - p\| + \alpha_n\|Tz_n - p\|] \\
&\leq \|x_{n+1}-p\| + \delta^3[(1-\alpha_n)\|w_n - p\| + \alpha_n\|z_n - p\|] \\
&= \|x_{n+1}-p\| + \delta^3[(1-\alpha_n)\|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\| \\
&\quad +\alpha_n\|T((1-\beta_n)w_n + \beta_n Tw_n) - p\|] \\
&\leq \|x_{n+1}-p\| + \delta^4[(1-\alpha_n)\|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\
&\quad +\alpha_n\|(1-\beta_n)w_n + \beta_n Tw_n - p\|] \\
&\leq \|x_{n+1}-p\| + \delta^4[(1-\alpha_n)[(1-\gamma_n)\|x_n - p\| + \gamma_n\delta\|x_n - p\|] \\
&\quad +\alpha_n[(1-\beta_n)\|w_n - p\| + \beta_n\delta\|w_n - p\|]] \\
&= \|x_{n+1}-p\| + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\|w_n - p\|] \\
&= \|x_{n+1}-p\| + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\|] \\
&\leq \|x_{n+1}-p\| + \delta^4[(1-\alpha_n)(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\quad +\alpha_n(1-\beta_n(1-\delta))\delta(1-\gamma_n(1-\delta))\|x_n - p\|] \\
&= \|x_{n+1}-p\| + \delta^4[((1-\alpha_n)(1-\gamma_n(1-\delta)) \\
&\quad +\alpha_n\delta(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta)))\|x_n - p\|]. \qquad (6.2)
\end{aligned}$$

Taking limit n → ∞, we obtain

$$\lim_{n\to\infty} \epsilon_n = 0.$$

7 Application in Solution of Nonlinear Integral Equation

Consider the following nonlinear integral equation:

$$x(t) = y(t) + \gamma \int_a^b f(t,s) g(t, x(s)) ds \qquad (7.1)$$

where γ ∈ [0, 1), f : [a, b] × [a, b] → R^+ , g : [a, b] × R → R and y : [a, b] → R all are continuous function. Let X = C([a, b], R) be the space of all continuous real valued function defined on [a, b] with ordered relation ≤ in X. Defined as for x, y ∈ R, x ≤ y if and only if x(s) ≤ y(s) for all s ∈ [a, b]. We defined ‖.‖ : X × X → [0, ∞) by

$$\|x - y\| = \sup_{s\in[a,b]} |x(s) - y(s)|.$$

Theorem 7.1. Let X = C([a, b], R) and T : X → X defined by

$$Tx(t) = y(t) + \gamma \int_a^b f(t,s) g(t, x(s)) ds \qquad (7.2)$$

where γ ∈ [0, 1), f : [a, b] × [a, b] → R^+ , g : [a, b] × R → R and y : [a, b] → R all are continuous function. Let X = C([a, b], R) be the space of all continuous real-valued functions defined on [a, b]. Suppose that the following conditions are satisfied :

(1) There exists a continuous mapping F : X × X → [0, ∞) such that

$$|g(s, x(s)) - g(s, y(s))| \leq F(x, y)|x(s) - y(s)|$$

for all x, y ∈ X and s ∈ [a, b].

(2) There exist ξ ∈ [0, 1] such that $\int_a^b f(t,s) F(x,y) ds \leq \xi$.

(3) γξ ≤ 1.

Let $\{x_n\}$ be a sequence in X defined by New-iteration (3.1). Then the sequence $\{x_n\}$ converges to a point of F (T) which is the solution of integral equation (7.1).

Proof. We suppose that x ≤ y, so

$$\sup\{|y(s) - x(s): s \in [a,b]\} \geq \sup\{|Tx(s) - x(s)|: s \in [a,b]\}$$
$$\Rightarrow \|Tx - x\| \leq \|y - x\|$$
$$\lambda\|Tx - x\| \leq \|Tx - x\| \leq \|y - x\|$$
$$\lambda\|Tx - x\| \leq \|y - x\|.$$

where λ ∈ [0, 1), thus we have

$$|Ty(s) - Tx(s)| = |y(t) + \gamma \int_a^b f(t,s) g(t, y(s)) ds - y(t) - \gamma \int_a^b f(t,s) g(t, x(s)) ds|$$
$$= |\gamma \int_a^b f(t,s)(g(t, y(s)) - g(t, x(s))) ds|$$
$$\leq \gamma \int_a^b F(x,y)|y(s) - x(s)| ds$$
$$\sup_{s\in[a,b]} |Ty(s) - Tx(s)| \leq \sup_{s\in[a,b]} |y(s) - x(s)| \gamma \int_a^b f(t,s) F(x,y) ds$$
$$\leq \|y - x\|\gamma\xi$$
$$\leq \|y - x\|.$$

Thus, we have

$$\lambda\|Tx - x\| \leq \|x - y\|$$
$$\Rightarrow \|Tx - Ty\| \leq \|x - y\|$$
$$\leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|.$$

Clearly, T satisfies condition (C_λ) and by Proposition 2.7, T is a generalized (α, β) nonexpansive type 1 mapping. Since T is generalized (α, β) - nonexpansive type 1 mapping and satisfying all conditions of Theorem 2.4, such as F (T) ≠ ϕ and the $\{x_n\}$ converges to a fixed point of T. This point will be the solution of the integral equation (7.1).

8 CONCLUSION

The main contribution of our work:

1. We introduced a four-step iteration scheme for approximating fixed points of generalized (α, β)-nonexpansive type 1 mapping.

2. We have established strong and weak convergence theorems for the new iteration process in the generalized (α, β)-nonexpansive type 1 mapping in uniformly convex Banach spaces.
3. We have proved convergence and stability results for contractive-like mapping in uniformly convex Banach space.
4. We provide numerical examples and show that this scheme is faster than Agarwal, Abbas, Thakur, Ullah, and S^{**} -iteration schemes.
5. We used the new iteration method to find the solution of a nonlinear integral equation.

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